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Research Paper

AN ITERATIVE ALGORITHM FOR SPLIT EQUALITY FIXED POINT AND NULL POINT PROBLEM OF LIPSCHITZIAN QUASI-PSEUDOCONTRACTIVE MAPPINGS

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Abstract. We introduce an iterative algorithm for split equality fixed point and null point problem (SEFPNPP) for Lipschitzian quasi-pseudocontractive mappings and maximal monotone operators which includes split equality feasibility problem, split equality fixed problem, split equality null point problem and other problems related to fixed point problem. Moreover, we establish strong convergence results in real Hilbert spaces under some suitable conditions and reduce our main result to abovementioned problems. Finally, we apply the study to split equality feasibility problem (SEFP), split equality equilibrium problem (SEEP), split equality variational inequality problem (SEVIP) and split equality optimization problem (SEOP). The results presented in the paper extend and improve many recent results.

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1. Introduction

Let C and Q be closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Consider two bounded linear operators $A : H_1 \to H_3$ and $B : H_2 \to H_3$, where H_3 is another real Hilbert space. The split equality feasibility problem consists of finding two points $x \in C$ and $y \in Q$ such that $Ax = By$. Split equality fixed problem allows asymmetric and partial relations between the variables x and y, and covers many problems such as decomposition methods for partial differential equations, applications in game theory, and intensity-modulated radiation therapy. These broad applications caught the attention of many researchers, and eventually leading to various research output for the split equality feasibility problem, (see for example [1, 2, 19, 28, 29, 30, 32]).

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Let C be a nonempty subset of a real Hilbert space H. A mapping $T : C \longrightarrow C$ is said to be **nonexpansive** if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C. \tag{1.1}$$

A mapping $T : C \longrightarrow C$ is said to be **quasi-nonexpansive** if $F(T) \neq \emptyset$ such that

$$\|Tx - p\| \leq \|x - p\|, \ \forall x \in C, \ p \in F(T). \tag{1.2}$$

A mapping $T : C \longrightarrow C$ is said to be **strictly quasi-nonexpansive** if $F(T) \neq \emptyset$ such that

$$\|Tx - p\| < \|x - p\|, \ \forall x \notin F(T), \ p \in F(T). \tag{1.3}$$

A mapping $T : C \longrightarrow C$ is said to be **strongly quasi-nonexpansive** if T is quasi-nonexpansive and

$$x_n - Tx_n \longrightarrow 0 \tag{1.4}$$

whenever $\{x_n\}$ is a bounded sequence in H and $\|x_n - p\| - \|Tx_n - p\| \longrightarrow 0$ for some $p \in F(T)$ and $n \geq 1$.
A mapping $T : C \longrightarrow C$ is said to be **firmly nonexpansive if**

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \ \forall x, y \in C. \tag{1.5}$$

A mapping $T : C \longrightarrow C$ is said to be **firmly quasi-nonexpansive** if $F(T) \neq \emptyset$ such that

$$\|Tx - p\|^2 \leq \|x - p\|^2 - \|x - Tx\|^2, \ \forall x \in C, \ p \in F(T). \tag{1.6}$$

A mapping $T : C \longrightarrow C$ is said to be $k-$**strictly pseudocontractive** if there exists a $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \ \forall x, y \in C. \tag{1.7}$$

If $k = 1$ in (1.7), then T is called a **pseudocontractive mapping**.
A mapping $T : C \longrightarrow C$ is said to be **demicontractive** if $F(T) \neq \emptyset$ and there exists a $k \in [0, 1)$ such that

$$\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2, \ \forall x \in C, \ p \in F(T). \tag{1.8}$$

A mapping $T : C \longrightarrow C$ is said to be **quasi-pseudocontractive** (see [31]) if

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \|x - Tx\|^2, \ \forall x \in C, \ p \in F(T). \tag{1.9}$$

Remark 1.1. We can observe that the class of quasi-pseudocontractive operators includes the class of operators defined in equations (1.1) - (1.8).

Let H_1 and H_2 be real Hilbert spaces and C and Q be nonempty closed and convex subsets of H_1 and H_2 respectively. The split feasibility problem (SFP) is formulated as: to find

$$x \in C \ \text{ such that } \ Ax \in Q \tag{1.10}$$

where $A : H_1 \to H_2$ is a bounded linear operator.

Censor and Elfving [9] first introduced the SFP in finite-dimensional Hilbert spaces for modeling inverse problems that arise from phase retrievals and in medical image reconstruction (see also [7]). It has been found that the SFP can also be used in various disciplines such as image restoration, computer tomography, and radiation therapy treatment planning [8, 10, 11]. The SFP in an infinite-dimensional real Hilbert space can be found in [7, 10, 12, 13, 27, 29, 32].

Moudafi [20, 22, 21] introduced the following split equality feasibility problem (SEFP) to find:

$$x \in C, y \in Q \text{ such that } Ax = By, \tag{1.11}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators. If $B = I$ (identity mapping on H_2) and $H_3 = H_2$, then (1.11) reduces to (1.10).

In order to solve split equality feasibility problem (1.11), Moudafi [20] introduced the following simultaneous iterative method:

$$\begin{array}{rcl} x_{n+1} & = & P_C(x_n - \gamma A^*(Ax_n - By_n)) \\ y_{n+1} & = & P_Q(y_n + \beta B^*(Ax_n - By_n)) \end{array} \quad n \in \mathbb{N} \tag{1.12}$$

where P_C is the metric projection of H onto C, P_Q is the metric projection of H onto Q, A^* is the adjoint of A, B^* is the adjoint of B and $\gamma > 0$, and under suitable conditions, he proved the weak convergence of the sequence $\{(x_n, y_n)\}$ to a solution of (1.11) in Hilbert spaces.

In order to avoid using the projection, recently, Moudafi [13] introduced and studied the following problem: Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be nonlinear operators such that $F(T) \neq \emptyset$ and $F(S) \neq \emptyset$, where $F(T)$ and $F(S)$ denote the sets of fixed points of T and S respectively. If $C = F(T)$ and $Q = F(S)$, then split equality feasibility problem (1.11) reduces to

$$x \in F(T), y \in F(S) \text{ such that } Ax = By, \tag{1.13}$$

which is called a split equality fixed point problem (SEFPP).

Moudafi [21] proposed the following iterative algorithm for finding a solution of SEFPP (1.13):

$$\begin{array}{rcl} x_{n+1} & = & T(x_n - \gamma_n A^*(Ax_n - By_n)) \\ y_{n+1} & = & S(y_n + \beta_n B^*(Ax_n - By_n)) \end{array} \quad n \in \mathbb{N}. \tag{1.14}$$

He also studied the weak convergence of the sequences generated by scheme (1.14) under the condition that T and S are firmly quasi-nonexpansive mappings.

Che and Li [16] proposed the following iterative algorithm for finding a solution of SEFPP (1.13):

$$\begin{array}{rcl} u_n & = & x_n - \gamma_n A^*(Ax_n - By_n) \\ x_{n+1} & = & \alpha_n x_n + (1 - \alpha_n)Tu_n \\ v_n & = & y_n + \beta_n B^*(Ax_n - By_n) \\ y_{n+1} & = & \alpha_n y_n + (1 - \alpha_n)Sv_n \end{array} \quad n \in \mathbb{N}. \tag{1.15}$$

They also established the weak convergence of the scheme (1.15) under the condition that the operators T and S are quasi-nonexpansive mappings.

Chang, Wang and Qin [14] proposed the following iterative algorithm for finding a solution of SEFPP (1.13):

$$\begin{array}{rcl} u_n & = & x_n - \gamma_n A^*(Ax_n - By_n) \\ x_{n+1} & = & \alpha_n x_n + (1-\alpha_n)((1-\xi)I + \xi T((1-\eta)I + \eta T))u_n \\ v_n & = & y_n + \beta_n B^*(Ax_n - By_n) \\ y_{n+1} & = & \alpha_n y_n + (1-\alpha_n)((1-\xi)I + \xi S((1-\eta)I + \eta S))v_n \end{array} \quad n \in \mathbb{N}. \tag{1.16}$$

They established the weak convergence of the scheme (1.16) under the condition that the operators T and S are quasi-pseudocontractive mappings.

Boikanyo and Zegeye [6] proposed the following iterative algorithm for finding a solution of SEFPP (1.13):

$$\begin{array}{rcl} u_n & = & P_C[x_n - \gamma_n A^*(Ax_n - By_n)] \\ x_{n+1} & = & \alpha_n u + (1-\alpha_n)((1-\xi)I + \xi T((1-\eta)I + \eta T))u_n \\ v_n & = & P_D[y_n - \gamma_n B^*(Ax_n - By_n)] \\ y_{n+1} & = & \alpha_n v + (1-\alpha_n)((1-\xi)I + \xi S((1-\eta)I + \eta S))v_n \end{array} \quad n \in \mathbb{N}. \tag{1.17}$$

They also established the strong convergence of the scheme (1.17) under the condition that the operators T and S are quasi-pseudocontractive mappings.

Motivated by the above works, we propose a new iterative algorithm called Halpern-type algorithm for the class of quasi-pseudocontractive mappings and maximal monotone operators that always converge strongly to the solution of the split equality fixed point and null point problem (SEFPNPP). It is known that the class of quasi-pseudocontractive mappings is more general than the class of quasi-contractive mappings, directed mappings, and demicontractive mappings. Moreover, strong convergence is more desirable than weak convergence and we obtain our result without additional conditions on the operators. Also, the implementation of the iterative algorithm does not require the calculation or estimation of the operator norms $\|A\|$ and $\|B\|$ which may at times be as difficult as solving the original problem itself. Hence, our results provide a unified framework for the study of the split equality fixed point and null point problem.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let C be a nonempty closed convex subset of H. The notation $x_n \to x$ denotes that the sequence $\{x_n\}$ converges strongly to x. Similarly, $x_n \rightharpoonup x$ will mean weak convergence.

For any $x \in H$, there exists a unique point $P_C x \in C$ such that

$$\|x - P_C x\| \leq \|x - y\|, \ \forall y \in C. \tag{2.1}$$

P_C is called the metric projection of H onto C. Note that P_C is a nonexpansive mapping of H onto C. For $x \in H$ and $z \in C$, we have

$$z = P_C x \Leftrightarrow \langle z - y, x - z \rangle \geq 0, \ \text{for every } y \in C. \tag{2.2}$$

In [6], it was shown that if H_1, H_2 are real Hilbert spaces, then $H := H_1 \times H_2$ is also a real Hilbert space with inner product

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle, \ \ \forall (x_1, y_1), (x_2, y_2) \in H_1 \times H_2$$

such that

$$(x_n, y_n) \rightharpoonup (x^*, y^*) \ \text{implies that} \ x_n \rightharpoonup x^* \ \text{and} \ y_n \rightharpoonup y^* \tag{2.3}$$

Moreover, if C is a nonempty, closed, and convex subset of H, $(u, v) \in H$ and $(u^*, v^*) = P_C(u, v)$, then from inequality (2.2), we obtain that

$$\langle (u^*, v^*) - (x, y), (u, v) - (u^*, v^*) \rangle \geq 0, \ \text{for every} \ y \in C, \ \ \forall (x, y) \in H. \tag{2.4}$$

Given a positive constant α, a mapping $A : C \to H$ is said to be $\alpha-$inverse strongly monotone if

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2 \ \forall \ x, y \in C. \tag{2.5}$$

For $\overline{\gamma} > 0$, a mapping A on H is called $\overline{\gamma}-$strongly monotone if

$$\langle x - y, Ax - Ay \rangle \geq \overline{\gamma} \|x - y\|^2 \ \ \forall \ x, y \in H. \tag{2.6}$$

Taking $L > 0$, a mapping A on H is said to be $L-$Lipschitzian continuous if

$$\|Ax - Ay\| \leq L \|x - y\|, \ \forall \ x, y \in H. \tag{2.7}$$

It can be seen that A is $\frac{\overline{\gamma}}{L^2}-$inverse strongly monotone whenever A is $\overline{\gamma}-$strongly monotone and $L-$Lipschitzian continuous.

Let B be a mapping of H into 2^H. The effective domain of B is denoted by $dom(B)$, that is, $dom(B) = \{x \in H : Bx \neq \emptyset\}$. A multivalued mapping B is said to be monotone if

$$\langle x - y, u - v \rangle \geq 0 \ \forall \ x, y \in dom(B), \ u \in Bx, \ v \in By \tag{2.8}$$

A monotone operator B is said to be maximal if its graph is not properly contained in the graph of any other monotone operator. For a maximal monotone operator B on H and $r > 0$, the operator

$$J_r = (I + rB)^{-1} : H \to \ \text{dom}(B) \tag{2.9}$$

is called the resolvent of B for r. It is known that J_r is firmly nonexpansive.

An operator h is called averaged (see [3]) if there exists a nonexpansive operator $N : D \to H$ and a number $\alpha \in (0, 1)$ such that

$$h = (1 - \alpha)I + \alpha N \tag{2.10}$$

where I is the identity operator.

Definition 2.1. Let $T : H \to H$, $I - T$ is called demi-closed at zero, if for any sequence $\{x_n\} \subset H$ and $x \in H$, we have $x_n \rightharpoonup x$ and $(I - T)x_n \to 0$, then $x \in Fix(T)$.

Lemma 2.2. [33] *Let H be a real Hilbert space, C a closed convex subset of H. Let $T : C \to C$ be a continuous pseudocontractive mapping. Then*
(i) $F(T)$ is a closed convex subset of C,
(ii) $(I - T)$ is demi-closed at zero.

Theorem 2.3. [18] *Let $T : H \to H$ be a $\alpha-$attracting quasi-nonexpansive operator where $\alpha > 0$ and $S : H \to H$ a strongly quasi-nonexpansive operator. Suppose that $F(T) \cap F(S) \neq \emptyset$. Then*
(i) Both TS and ST are strongly quasi-nonexpansive,
(ii) If $I - T$ and $I - S$ are demi-closed at zero, then $I - TS$ and $I - ST$ are also demi-closed at zero.

Lemma 2.4. [15] *Let $T : H \to H$ be a strictly quasi-nonexpansive operator and $S : H \to H$ a quasi-nonexpansive operator. Suppose that $F(T) \cap F(S) \neq \emptyset$. Then $F(TS) = F(ST) = F(T) \cap F(S)$.*

Lemma 2.5. [26]. *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\beta_n + \gamma_n, \ n \geq 0 \tag{2.11}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions: (i) $\{\alpha_n\} \subset [0, 1]$, $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$, (ii) $\limsup\limits_{n\to\infty} \beta_n \leq 0$, (iii) $\gamma_n \geq 0$, $\sum\limits_{n=1}^{\infty} \gamma_n < \infty$. Then $\lim\limits_{n\to\infty} s_n = 0$.

Lemma 2.6. [23] *Let H be a Hilbert space with inner product $\langle\cdot\rangle$ and norm $\|$ respectively. Then $\forall x, y \in H$,*

$$\begin{aligned} (i) \quad \|tx + (1 - t)y\|^2 &= t\|x\|^2 + (1 - t)\|y\|^2 \\ &\quad -t(1 - t)\|x - y\|^2, \ \forall t \in [0, 1]. \end{aligned} \tag{2.12}$$

$$(ii) \quad \|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 \tag{2.13}$$

3. Main Results

Theorem 3.1. *Let H_1 and H_2 be real Hilbert spaces. Let B_1 and B_2 be maximal monotone operators of H_1 into 2^{H_1} and H_2 into 2^{H_2} and $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ be resolvents of B_1 and B_2, respectively for $\lambda > 0$. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators, and $S : H_1 \to H_1$ be Lipschitzian quasi-pseudocontractive self maps of H_1 and $T : H_2 \to H_2$ be Lipschitzian quasi-pseudocontractive self maps of H_2 such that $(I - S)$ and $(I-T)$ are demi-closed at zero. If the solution set of SEFPNPP is nonempty (that is, $\Gamma = \{(x, y) : x \in F(S) \cap B_1^{-1}0, y \in F(T) \cap B_2^{-1}0, Ax = By\} \neq \emptyset$). Suppose that $x_0, x_1 \in H_1$ and $y_0, y_1 \in H_2$ are chosen arbitrarily. Let $\{(x_n, y_n)\}$ be the iterative sequence generated by*

$$\begin{aligned} x_{n+1} &= \beta_n x_0 + (1 - \beta_n)u_n \\ u_n &= \alpha_n x_n + (1 - \alpha_n)SJ_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n)) \\ y_{n+1} &= \beta_n y_0 + (1 - \beta_n)v_n \\ v_n &= \alpha_n y_n + (1 - \alpha_n)TJ_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n)) \end{aligned} \quad n \geq 1 \tag{3.1}$$

where the parameter γ and the sequences $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ satisfy the conditions: (i) $\gamma \in \left(0, \frac{2(1-\beta_n L^2)+\beta_n L^2(\|A\|^2+\|B\|^2)}{(1+2(L+1)^2)(\|A\|^2+\|B\|^2)}\right)$ (ii) $\sum_{n=1}^{\infty} \alpha_n < \infty$, (iii) $\lim\limits_{n\to\infty} \beta_n = 0$ and (iv) $\sum_{n=1}^{\infty} \beta_n = \infty$. Then,

(a) $\lim_{n\to\infty} \Phi_n(p,q)$ *exists for each* $(p,q) \in \Gamma$, $p \in F(S)\cap B_1^{-1}0, q \in F(T)\cap B_2^{-1}0, Ap = Bq,$

(b) $\lim_{n\to\infty} \|x_n - SJ_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n))\| = \lim_{n\to\infty} \|y_n - TJ_\lambda^{B_1}(y_n + \gamma B^*(Ax_n - By_n))\| = 0,$

(c) $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $(p,q) \in \Gamma$, $p \in F(S)\cap B_1^{-1}0, q \in F(T)\cap B_2^{-1}0, Ap = Bq.$

Proof : We use Lemma 2.6 (see also [23]) and the fact that S and T are $L-$Lipschitzians.

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2 \tag{3.2}$$

which holds $\forall x, y \in H_1$. Let $(p,q) \in \Gamma$, then using (3.1) and (3.2), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\beta_n x_0 + (1-\beta_n)u_n - p\|^2\\ &= \|\beta_n(x_0 - p) + (1-\beta_n)(u_n - p)\|^2\\ &= \beta_n\|x_0 - p\|^2 + (1-\beta_n)\|u_n - p\|^2 - \beta_n(1-\beta_n)\|u_n - x_0\|^2. \end{aligned}\tag{3.3}$$

$$\begin{aligned}\|u_n - p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)(SJ_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n)) - p)\|^2\\ &= \|\alpha_n(x_n - p) + (1-\alpha_n)(SJ_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n)) - p)\|^2\\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|SJ_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n)) - p\|^2\\ &\quad -\alpha_n(1-\alpha_n)\|SJ_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n)) - x_n\|^2. \end{aligned}\tag{3.4}$$

Substitute equation (3.4) into (3.3) to obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \beta_n\|x_0 - p\|^2 + (1-\beta_n)\{\alpha_n\|x_n - p\|^2\\ &\quad +(1-\alpha_n)\|SJ_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n)) - p\|^2\\ &\quad -\alpha_n(1-\alpha_n)\|SJ_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n)) - x_n\|^2\}\\ &\quad -\beta_n(1-\beta_n)\|u_n - x_0\|^2\\ &= \beta_n\|x_0 - p\|^2 + \alpha_n(1-\beta_n)\|x_n - p\|^2\\ &\quad +(1-\alpha_n)(1-\beta_n)\|SJ_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n)) - p\|^2\\ &\quad -\alpha_n(1-\alpha_n)(1-\beta_n)\|SJ_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n)) - x_n\|^2\\ &\quad -\beta_n(1-\beta_n)\|u_n - x_0\|^2. \end{aligned}\tag{3.5}$$

Since S is Lipschitzian quasi-pseudocontractive and $J_\lambda^{B_1}$ is nonexpansive, we have

$$\begin{aligned}\|SJ_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n)) - p\|^2 &\leq \|J_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n)) - p\|^2\\ &\quad +\|SJ_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n)) - J_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n))\|^2\\ &\leq \|x_n - p\|^2 + \gamma^2\|A^*(Ax_n - By_n)\|^2 - 2\gamma\langle x_n - p, A^*(Ax_n - By_n)\rangle\\ &\quad +\|SJ_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n)) - J_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n))\|^2\\ &= \|x_n - p\|^2 + \gamma^2\|A^*(Ax_n - By_n)\|^2 - 2\gamma\langle Ax_n - Ap, Ax_n - By_n\rangle\\ &\quad +\|SJ_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n)) - J_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n))\|^2. \end{aligned}\tag{3.6}$$

$$\begin{aligned} \|A^*(Ax_n - By_n)\|^2 &= \langle A^*(Ax_n - By_n), A^*(Ax_n - By_n)\rangle \\ &= \langle AA^*(Ax_n - By_n), Ax_n - By_n\rangle \\ &= \|A\|^2\|Ax_n - By_n\|^2. \end{aligned} \tag{3.7}$$

$$\begin{aligned} \|SJ_\lambda^{B_1}(x_n &- \gamma A^*(Ax_n - By_n)) - J_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n))\| \\ &= \|SJ_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n)) - p + p - J_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n))\| \\ &\leq L\|J_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n)) - p\| + \|J_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n)) - p\| \\ &\leq (L+1)\|x_n - p - \gamma A^*(Ax_n - By_n)\| \\ &\leq (L+1)\|x_n - p\| + \gamma(L+1)\|A^*(Ax_n - By_n)\| \end{aligned} \tag{3.8}$$

Substitute equation (3.7) into (3.8), we have

$$\begin{aligned} \|SJ_\lambda^{B_1}(x_n &- \gamma A^*(Ax_n - By_n)) - J_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n))\| \\ &\leq (L+1)\|x_n - p\| + \gamma(L+1)\|A\|\|Ax_n - By_n\| \end{aligned} \tag{3.9}$$

therefore,

$$\begin{aligned} \|SJ_\lambda^{B_1}(x_n &- \gamma A^*(Ax_n - By_n)) - J_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n))\|^2 \\ &\leq ((L+1)\|x_n - p\| + \gamma(L+1)\|A\|\|Ax_n - By_n\|)^2 \\ &\leq (L+1)^2\|x_n - p\|^2 + \gamma^2(L+1)^2\|A\|^2\|Ax_n - By_n\|^2 \\ &\quad +(L+1)^2\|x_n - p\|^2 + \gamma^2(L+1)^2\|A\|\|Ax_n - By_n\|^2 \\ &= 2(L+1)^2\|x_n - p\|^2 + 2\gamma^2(L+1)^2\|A\|^2\|Ax_n - By_n\|^2. \end{aligned} \tag{3.10}$$

Substitute equations (3.7) and (3.10) into (3.6) to obtain

$$\begin{aligned} \|SJ_\lambda^{B_1}(x_n &- \gamma A^*(Ax_n - By_n) - p\|^2 \leq \|x_n - p\|^2 + \gamma^2\|A\|^2\|Ax_n - By_n\|^2 \\ &\quad -2\gamma\langle Ax_n - Ap, Ax_n - By_n\rangle + 2(L+1)^2\|x_n - p\|^2 \\ &\quad +2\gamma(L+1)^2\|A\|^2\|Ax_n - By_n\|^2 \\ &= [1 + 2(L+1)^2]\|x_n - p\|^2 \\ &\quad +[1 + 2(L+1)^2]\gamma^2\|A\|^2\|Ax_n - By_n\|^2 \\ &\quad -2\gamma\langle Ax_n - Ap, Ax_n - By_n\rangle. \end{aligned} \tag{3.11}$$

Substitute equation (3.11) into (3.5) to obtain

$$\begin{aligned}\|x_{n+1} \;-\; & p\|^2 \leq \beta_n\|x_0-p\|^2+\alpha_n(1-\beta_n)\|x_n-p\|^2\\ & +(1-\alpha_n)(1-\beta_n)\{[1+2(L+1)^2\|x_n-p\|^2\\ & +[1+2(L+1)^2]\gamma^2\|A\|^2\|Ax_n-By_n\|^2\\ & -2\gamma\langle Ax_n-Ap, Ax_n-By_n\rangle\}\\ & -\alpha_n(1-\alpha_n)(1-\beta_n)\|SJ_\lambda^{B_1}(x_n-\gamma A^*(Ax_n-By_n))-x_n\|^2\\ & -\beta_n(1-\beta_n)\|u_n-x_0\|^2\\ = \; & \beta_n\|x_0-p\|^2+\alpha_n(1-\beta_n)\|x_n-p\|^2\\ & +(1-\alpha_n)(1-\beta_n)[1+2(L+1)^2]\|x_n-p\|^2\\ & +(1-\alpha_n)(1-\beta_n)[1+2(L+1)^2]\gamma^2\|A\|^2\|Ax_n-By_n\|^2\\ & -2\gamma(1-\alpha_n)(1-\beta_n)\langle Ax_n-Ap, Ax_n-By_n\rangle\\ & -\alpha_n(1-\alpha_n)(1-\beta_n)\|SJ_\lambda^{B_1}(x_n-\gamma A^*(Ax_n-By_n))-x_n\|^2\\ & -\beta_n(1-\beta_n)\|u_n-x_0\|^2. \qquad (3.12)\end{aligned}$$

$$\begin{aligned}\|u_n \;-\; & x_0\|^2=\|\alpha_nx_n+(1-\alpha_n)SJ_\lambda^{B_1}(x_n-\gamma A^*(Ax_n-By_n))-x_0\|^2\\ = \; & \|\alpha_n(x_n-x_0)+(1-\alpha_n)(SJ_\lambda^{B_1}(x_n-\gamma A^*(Ax_n-By_n))-x_0)\|^2\\ \leq \; & \alpha_n\|x_n-x_0\|^2+(1-\alpha_n)L^2\|x_n-x_0\|^2+(1-\alpha_n)L^2\gamma\|A^*(Ax_n-By_n)\|^2\\ & -2\gamma(1-\alpha_n)L^2\langle x_n-x_0, A^*(Ax_n-By_n)\rangle\\ & -\alpha_n(1-\alpha_n)\|SJ_\lambda^{B_1}(x_n-\gamma A^*(Ax_n-By_n))-x_n\|^2\end{aligned}$$

$$\begin{aligned}= \; & [\alpha_n+(1-\alpha_n)L^2]\|x_n-x_0\|^2+(1-\alpha_n)L^2\gamma\|A\|^2\|Ax_n-By_n\|^2\\ & -2\gamma(1-\alpha_n)L^2\langle Ax_n-Ax_0, Ax_n-By_n\rangle\\ & -\alpha_n(1-\alpha_n)\|SJ_\lambda^{B_1}(x_n-\gamma A^*(Ax-Bx_n))-x_n\|^2. \qquad (3.13)\end{aligned}$$

Substitute equation (3.13) into (3.12) to obtain

$$\begin{aligned}\|x_{n+1} \;-\; & p\|^2 \leq \beta_n\|x_0-p\|^2+\alpha_n(1-\beta_n)\|x_n-p\|^2\\ & +(1-\alpha_n)(1-\beta_n)[1+2(L+1)^2]\|x_n-p\|^2\\ & +(1-\alpha_n)(1-\beta_n)[1+2(L+1)^2]\gamma^2\|A\|^2\|Ax_n-By_n\|^2\\ & -2\gamma(1-\alpha_n)(1-\beta_n)\langle Ax_n-Ap, Ax_n-By_n\rangle\\ & -\alpha_n(1-\alpha_n)(1-\beta_n)\|SJ_\lambda^{B_1}(x_n-\gamma A^*(Ax_n-By_n))-x_n\|^2\\ & -\beta_n(1-\beta_n)\{[\alpha_n+(1-\alpha_n)L^2]\|x_n-x_0\|^2\\ & +(1-\alpha_n)L^2\gamma\|A\|^2\|Ax_n-By_n\|^2\\ & -2\gamma(1-\alpha_n)L^2\langle Ax_n-Ax_0, Ax_n-By_n\rangle\\ & -\alpha_n(1-\alpha_n)\|SJ_\lambda^{B_1}(x_n-\gamma A^*(Ax_n-By_n))-x_n\|^2\}\end{aligned}$$

$$
\begin{aligned}
&= [1-\beta_n+2(1-\alpha_n)(1-\beta_n)(L+1)^2]\|x_n-p\|^2 \\
&\quad +(1-\alpha_n)(1-\beta_n)[1+2(L+1)^2]\gamma^2\|A\|^2\|Ax_n-By_n\|^2 \\
&\quad -\beta_n(1-\beta_n)(1-\alpha_n)L^2\gamma\|A\|^2\|Ax_n-By_n\|^2 \\
&\quad -2\gamma(1-\alpha_n)(1-\beta_n)\langle Ax_n-Ap, Ax_n-By_n\rangle \\
&\quad +2\gamma\beta_n(1-\beta_n)(1-\alpha_n)L^2\langle Ax_n-Ax_0, Ax_n-By_n\rangle \\
&\quad -\alpha_n(1-\alpha_n)(1-\beta_n)^2\|SJ_\lambda^{B_1}(x_n-\gamma A^*(Ax_n-By_n))-x_n\|^2 \\
&\quad -\beta_n(1-\beta_n)[\alpha_n+(1-\alpha_n)L^2]\|x_n-x_0\|^2 \\
&\quad +\beta_n\|x_0-p\|^2. \qquad (3.14)
\end{aligned}
$$

$$
\begin{aligned}
\|y_{n+1} - q\|^2 &= \|\beta_n y_0+(1-\beta_n)v_n-q\|^2 \\
&= \|\beta_n(y_0-q)+(1-\beta_n)(v_n-q)\|^2 \\
&= \beta_n\|y_0-p\|^2+(1-\beta_n)\|v_n-p\|^2-\beta_n(1-\beta_n)\|v_n-y_0\|^2. \qquad (3.15)
\end{aligned}
$$

$$
\begin{aligned}
\|v_n - q\|^2 &= \|\alpha_n y_n+(1-\alpha_n)(TJ_\lambda^{B_2}(y_n+\gamma B^*(Ax_n-By_n))-q)\|^2 \\
&= \|\alpha_n(y_n-q)+(1-\alpha_n)(TJ_\lambda^{B_2}(x_n+\gamma B^*(Ax_n-By_n))-q)\|^2 \\
&= \alpha_n\|y_n-q\|^2+(1-\alpha_n)\|TJ_\lambda^{B_2}(y_n+\gamma B^*(Ax_n-By_n))-q\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|TJ_\lambda^{B_2}(y_n+\gamma B^*(Ax_n-By_n))-y_n\|^2. \qquad (3.16)
\end{aligned}
$$

Substitute equation (3.16) into (3.15) to obtain

$$
\begin{aligned}
\|y_{n+1} - q\|^2 &= \beta_n\|y_0-q\|^2+(1-\beta_n)\{\alpha_n\|y_n-q\|^2 \\
&\quad +(1-\alpha_n)\|TJ_\lambda^{B_2}(y_n+\gamma B^*(Ax_n-By_n))-q\|^2 \\
&\quad -\alpha_n(1-\alpha_n\|TJ_\lambda^{B_2}(y_n+\gamma B^*(Ax_n-By_n))-y_n\|^2\}-\beta_n(1-\beta_n\|y_n-y_0\|^2 \\
&= \beta_n\|y_0-q\|^2+\alpha_n(1-\beta_n)\|y_n-q\|^2 \\
&\quad +(1-\alpha_n)(1-\beta_n)\|TJ_\lambda^{B_2}(y_n+\gamma B^*(Ax_n-By_n))-p\|^2 \\
&\quad -\alpha_n(1-\alpha_n)(1-\beta_n)\|TJ_\lambda^{B_2}(y_n+\gamma B^*(Ax_n-By_n))-y_n\|^2 \\
&\quad -\beta_n(1-\beta_n)\|v_n-y_0\|^2. \qquad (3.17)
\end{aligned}
$$

Since T is Lipschitzian quasi-pseudocontractive and $J_\lambda^{B_2}$ is nonexpansive, we have

$$\begin{aligned}
\|TJ_\lambda^{B_2}(y_n &+ \gamma B^*(Ax_n - By_n)) - q\|^2 \leq \|J_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n)) - q\|^2 \\
&\quad + \|TJ_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n)) - J_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n))\|^2 \\
&\leq \|y_n - q + \gamma B^*(Ax_n - By_n))\|^2 \\
&\quad + \|TJ_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n)) - J_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n))\|^2 \\
&\leq \|y_n - q\|^2 + \gamma^2\|B^*(Ax_n - By_n)\|^2 + 2\gamma\langle y_n - q, B^*(Ax_n - By_n)\rangle \\
&\quad + \|TJ_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n)) - J_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n))\|^2 \\
&= \|y_n - q\|^2 + \gamma^2\|B^*(Ax_n - By_n)\|^2 + 2\gamma\langle By_n - Bq, Ax_n - By_n\rangle \\
&\quad + \|TJ_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n)) - J_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n))\|^2. \qquad (3.18)
\end{aligned}$$

$$\begin{aligned}
\|B^*(Ax_n - By_n)\|^2 &= \langle B^*(Ax_n - By_n), B^*(Ax_n - By_n)\rangle \\
&= \langle BB^*(Ax_n - By_n), Ax_n - By_n\rangle \\
&= \|B\|^2\|Ax_n - By_n\|^2. \qquad (3.19)
\end{aligned}$$

$$\begin{aligned}
\|TJ_\lambda^{B_2}(y_n &+ \gamma B^*(Ax_n - By_n)) - J_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n))\| \\
&= \|TJ_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n)) - q + q - J_\lambda^{B_1}(y_n + \gamma B^*(Ax_n - By_n))\| \\
&\leq \|TJ_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n)) - q\| + \|J_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n)) - q\| \\
&\leq L\|J_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n)) - q\| + \|J_\lambda^{B_1}(y_n + \gamma B^*(Ax_n - By_n)) - p\| \\
&\leq (L+1)\|y_n - q + \gamma B^*(Ax_n - By_n)\| \\
&\leq (L+1)\|y_n - q\| + \gamma(L+1)\|B^*(Ax_n - By_n)\|. \qquad (3.20)
\end{aligned}$$

Substitute equation (3.19) into (3.20), we have

$$\begin{aligned}
\|TJ_\lambda^{B_2}(y_n &+ \gamma B^*(Ax_n - By_n) - J_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n)\| \\
&\leq (L+1)\|y_n - q\| + \gamma(L+1)\|B\|\|Ax_n - By_n\| \qquad (3.21)
\end{aligned}$$

therefore,

$$\begin{aligned}
\|TJ_\lambda^{B_2}(y_n &+ \gamma B^*(Ax_n - By_n) - J_\lambda^{B_1}(y_n + \gamma B^*(Ax_n - By_n)\|^2 \\
&\leq 2(L+1)^2\|y_n - q\| + 2\gamma^2(L+1)^2\|B\|^2\|Ax_n - By_n\|^2. \qquad (3.22)
\end{aligned}$$

Substitute equations (3.19) and (3.22) into (3.18) to obtain

$$\begin{aligned}
\|TJ_\lambda^{B_2}(y_n &+ \gamma B^*(Ax_n - By_n) - q\|^2 \leq \|y_n - q\|^2 + \gamma^2\|B\|^2\|Ax_n - By_n\|^2 \\
&\quad + 2\gamma\langle By_n - Bq, Ax_n - By_n\rangle + 2(L+1)^2\|y_n - p\|^2 \\
&\quad + 2\gamma(L+1)^2\|B\|^2\|Ax_n - By_n\|^2 \\
&= [1 + 2(L+1)^2]\|y_n - q\|^2 \\
&\quad + [1 + 2(L+1)^2]\gamma^2\|B\|^2\|Ax_n - By_n\|^2 \\
&\quad + 2\gamma\langle By_n - Bq, Ax_n - By_n\rangle \qquad (3.23)
\end{aligned}$$

Substitute equation (3.23) into (3.17) to obtain

$$\begin{aligned}
\|y_{n+1} - p\|^2 &\leq \beta_n\|y_0 - q\|^2 + \alpha_n(1-\beta_n)\|y_n - q\|^2 \\
&+(1-\alpha_n)(1-\beta_n)\{[1+2(L+1)^2\|x_n - p\|^2 \\
&+[1+2(L+1)^2]\gamma^2\|B\|^2\|Ax_n - By_n\|^2 \\
&+2\gamma\langle By_n - Bq, Ax_n - By_n\rangle\} \\
&-\alpha_n(1-\alpha_n)(1-\beta_n)\|TJ_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n)) - y_n\|^2 \\
&-\beta_n(1-\beta_n)\|v_n - x_0\|^2 \\
&= \beta_n\|y_0 - q\|^2 + \alpha_n(1-\beta_n)\|y_n - q\|^2 \\
&+(1-\alpha_n)(1-\beta_n)[1+2(L+1)^2]\|y_n - q\|^2 \\
&+(1-\alpha_n)(1-\beta_n)[1+2(L+1)^2]\gamma^2\|B\|^2\|Ax_n - By_n\|^2 \\
&+2\gamma(1-\alpha_n)(1-\beta_n)\langle By_n - Bq, Ax_n - By_n\rangle \\
&-\alpha_n(1-\alpha_n)(1-\beta_n)\|TJ_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n)) - y_n\|^2 \\
&-\beta_n(1-\beta_n)\|v_n - y_0\|^2. \qquad (3.24)
\end{aligned}$$

$$\begin{aligned}
\|v_n - y_0\|^2 &= \|\alpha_n y_n + (1-\alpha_n)TJ_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n)) - y_0\|^2 \\
&= \|\alpha_n(y_n - y_0) + (1-\alpha_n)(TJ_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n)) - y_0)\|^2 \\
&\leq \alpha_n\|y_n - y_0\|^2 + (1-\alpha_n)L^2\|y_n - y_0\|^2 + (1-\alpha_n)L^2\gamma\|B^*(Ax_n - By_n)\|^2 \\
&+2\gamma(1-\alpha_n)L^2\langle y_n - y_0, B^*(Ax_n - By_n)\rangle \\
&-\alpha_n(1-\alpha_n)\|TJ_\lambda^{B_2}(y_n - \gamma B^*(Ax_n - By_n)) - y_n\|^2 \\
&= [\alpha_n + (1-\alpha_n)L^2]\|y_n - y_0\|^2 + (1-\alpha_n)L^2\gamma\|B\|^2\|Ax_n - By_n\|^2 \\
&+2\gamma(1-\alpha_n)L^2\langle By_n - By_0, Ax_n - By_n\rangle \\
&-\alpha_n(1-\alpha_n)\|TJ_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n)) - y_n\|^2. \qquad (3.25)
\end{aligned}$$

Substitute equation (3.25) into (3.24) to obtain

$$\begin{aligned}
\|y_{n+1} - q\|^2 &\leq \beta_n\|y_0 - q\|^2 + \alpha_n(1-\beta_n)\|y_n - q\|^2 \\
&+(1-\alpha_n)(1-\beta_n)[1+2(L+1)^2]\|y_n - q\|^2 \\
&+(1-\alpha_n)(1-\beta_n)[1+2(L+1)^2]\gamma^2\|B\|^2\|Ax_n - By_n\|^2 \\
&+2\gamma(1-\alpha_n)(1-\beta_n)\langle By_n - Bq, Ax_n - By_n\rangle \\
&-\alpha_n(1-\alpha_n)(1-\beta_n)\|TJ_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n)) - y_n\|^2 \\
&-\beta_n(1-\beta_n)\{[\alpha_n + (1-\alpha_n)L^2]\|x_n - x_0\|^2 \\
&+(1-\alpha_n)L^2\gamma\|B\|^2\|Ax_n - By_n\|^2 \\
&+2\gamma(1-\alpha_n)L^2\langle By_n - By_0, Ax_n - By_n\rangle \\
&-\alpha_n(1-\alpha_n)\|TJ_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n)) - y_n\|^2\}
\end{aligned}$$

$$
\begin{aligned}
= \quad & [1-\beta_n+2(1-\alpha_n)(1-\beta_n)(L+1)^2]\|y_n-q\|^2 \\
& +(1-\alpha_n)(1-\beta_n)[1+2(L+1)^2]\gamma^2\|B\|^2\|Ax_n-By_n\|^2 \\
& -\beta_n(1-\beta_n)(1-\alpha_n)L^2\gamma\|B\|^2\|Ax_n-By_n\|^2 \\
& +2\gamma(1-\alpha_n)(1-\beta_n)\langle By_n-Bq, Ax_n-By_n\rangle \\
& -2\gamma\beta_n(1-\beta_n)(1-\alpha_n)L^2\langle By_n-By_0, Ax_n-By_n\rangle \\
& -\alpha_n(1-\alpha_n)(1-\beta_n)^2\|TJ_\lambda^{B_2}(y_n+\gamma B^*(Ax_n-By_n))-y_n\|^2 \\
& -\beta_n(1-\beta_n)[\alpha_n+(1-\alpha_n)L^2]\|y_n-y_0\|^2 \\
& +\beta_n\|y_0-q\|^2. \qquad (3.26)
\end{aligned}
$$

$$
\begin{aligned}
& \|x_{n+1}-p\|^2+\|y_{n+1}-q\|^2 \\
& \leq [1-\beta_n+2(1-\alpha_n)(1-\beta_n)(L+1)^2][\|x_n-p\|^2+\|y_n-q\|^2] \\
& +(1-\alpha_n)(1-\beta_n)[1+2(L+1)^2](\gamma^2\|A\|^2+\gamma^2\|B\|^2)\|Ax_n-By_n\|^2 \\
& -\beta_n(1-\beta_n)(1-\alpha_n)L^2(\gamma\|A\|^2+\gamma\|B\|^2)\|Ax_n-By_n\|^2 \\
& -2\gamma(1-\alpha_n)(1-\beta_n)\langle Ax_n-Ap, Ax_n-By_n\rangle \\
& +2\gamma(1-\alpha_n)(1-\beta_n)\langle By_n-Bq, Ax_n-By_n\rangle \\
& +2\gamma\beta_n(1-\beta_n)(1-\alpha_n)L^2\langle Ax_n-Ax_0, Ax_n-By_n\rangle \\
& -2\gamma\beta_n(1-\beta_n)(1-\alpha_n)L^2\langle By_n-By_0, Ax_n-Bx_n\rangle \\
& -\alpha_n(1-\alpha_n)(1-\beta_n)^2(\|SJ_\lambda^{B_1}(x_n-\gamma A^*(Ax_n-By_n))-x_n\|^2 \\
& +\|TJ_\lambda^{B_2}(y_n+\gamma B^*(Ax_n-By_n))-y_n\|^2) \\
& -\beta_n(1-\beta_n)[\alpha_n+(1-\alpha_n)L^2](\|x_n-x_0\|^2+\|y_n-y_0\|^2) \\
& +\beta_n(\|x_0-p\|^2+\|y_0-q\|^2) \\
= \quad & [1-\beta_n+2(1-\alpha_n)(1-\beta_n)(L+1)^2][\|x_n-p\|^2+\|y_n-q\|^2] \\
& -(1-\alpha_n)(1-\beta_n)[2\gamma(1-\beta_nL^2)-\{(1+2(L+1)^2)\gamma \\
& -\beta_nL^2\}(\gamma\|A\|^2+\gamma\|B\|^2)]\|Ax_n-By_n\|^2 \\
& -2\gamma(1-\alpha_n)(1-\beta_n)(1-\beta_nL^2)\|Ax_n-By_n\|^2 \\
& -\alpha_n(1-\alpha_n)(1-\beta_n)^2(\|SJ_\lambda^{B_1}(x_n-\gamma A^*(Ax_n-By_n))-x_n\|^2 \\
& +\|TJ_\lambda^{B_2}(y_n+\gamma B^*(Ax_n-By_n))-y_n\|^2) \\
& -\beta_n(1-\beta_n)[\alpha_n+(1-\alpha_n)L^2](\|x_n-x_0\|^2+\|y_n-y_0\|^2) \\
& \qquad +\beta_n(\|x_0-p\|^2+\|y_0-q\|^2) \qquad (3.27)
\end{aligned}
$$

$$\begin{aligned}
&\Phi_{n+1}(p,q) \leq [1-\delta_n]\Phi_n(p,q) + \beta_n\Phi_0(p,q) \\
&-(1-\alpha_n)(1-\beta_n)[2\gamma(1-\beta_nL^2) - \{(1+2(L+1)^2)\gamma \\
&-\beta_nL^2\}(\gamma\|A\|^2+\gamma\|B\|^2)]\|Ax_n - By_n\|^2 \\
&-2\gamma(1-\alpha_n)(1-\beta_n)(1-\beta_nL^2)\|Ax_n - By_n\|^2 \\
&-\alpha_n(1-\alpha_n)(1-\beta_n)^2[\|SJ_\lambda^{B_1}(x_n-\gamma A^*(Ax_n-By_n)) - x_n\|^2 \\
&+\|TJ_\lambda^{B_2}(y_n+\gamma B^*(Ax_n-By_n)) - y_n\|^2] \\
&-\beta_n(1-\beta_n)[\alpha_n+(1-\alpha_n)L^2](\|x_n-x_0\|^2+\|y_n-y_0\|^2) \qquad (3.28)
\end{aligned}$$

where

$$\delta_n = \beta_n - 2(1-\alpha_n)(1-\beta_n)(L+1)^2.$$

By condition (iv) $\sum_{n=1}^{\infty}\beta_n = \infty$ and then $\sum_{n=1}^{\infty}\delta_n = \infty$. Hence from Lemma 2.5 that following $\lim_{n\to\infty}\Phi_n(p,q)$ exists, implies

$$\lim_{n\to\infty}\|x_n - p\| \text{ and } \lim_{n\to\infty}\|y_n - q\|. \qquad (3.29)$$

From equation (3.28)

$$\begin{aligned}
&(1-\alpha_n)(1-\beta_n)[2\gamma(1-\beta_nL^2) - \{(1+2(L+1)^2)\gamma \\
&-\beta_nL^2\}(\gamma\|A\|^2+\gamma\|B\|^2)]\|Ax_n - By_n\|^2 \\
&+\alpha_n(1-\alpha_n)(1-\beta_n)^2(\|SJ_\lambda^{B_1}(x_n-\gamma A^*(Ax_n-By_n)) - x_n\|^2 \\
&+\|TJ_\lambda^{B_2}(y_n+\gamma B^*(Ax_n-By_n)) - y_n\|^2) \\
\leq\ &\Phi_n(p,q) - \delta_n\Phi_n(p,q) + \beta_n\Phi_0(p,q) \\
&-\Phi_{n+1}(p,q) \to 0 \text{ (as } n\to\infty). \qquad (3.30)
\end{aligned}$$

Since $\gamma \in \left(0, \frac{2(1-\beta_nL^2)+\beta_nL^2(\|A\|^2+\|B\|^2)}{(1+2(L+1)^2)(\|A\|^2+\|B\|^2)}\right)$, this implies that

$$\lim_{n\to\infty}\|Ax_n - By_n\| = 0 \qquad (3.31)$$

$$\lim_{n\to\infty}\|SJ_\lambda^{B_1}(x_n-\gamma A^*(Ax_n-By_n)) - x_n\| = 0 \qquad (3.32)$$

$$\lim_{n\to\infty}\|TJ_\lambda^{B_2}(y_n+\gamma B^*(Ax_n-By_n)) - y_n\| = 0 \qquad (3.33)$$

Also,

$$\lim_{n\to\infty}\Phi_{n+1}(x_n,y_n) = 0, \qquad (3.34)$$

It follows from equations (3.1)

$$\begin{aligned}
&\|x_{n+1} - x_n\| = \|\beta_nx_0 + (1-\beta_n)y_n - x_n\| \\
=\ &\|\beta_n(x_0-x_n) + (1-\beta_n)(y_n-x_n)\| \\
=\ &\|\beta_n(x_0-x_n) + (1-\beta_n)(\alpha_nx_n + (1-\alpha_n)(SJ_\lambda^{B_1}(x_n-\gamma A^*(Ax_n-By_n)) - x_n)\| \\
=\ &\|\beta_n(x_0-x_n) + (1-\beta_n)(1-\alpha_n)(SJ_\lambda^{B_1}(x_n-\gamma A^*(Ax_n-By_n)) - x_n)\| \\
\leq\ &\beta_n\|x_0-p\| + \beta_n\|x_n-p\| \\
&+(1-\beta_n)(1-\alpha_n)\|SJ_\lambda^{B_1}(x_n-\gamma A^*(Ax_n-By_n)) - x_n\| \qquad (3.35)
\end{aligned}$$

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \|\beta_n y_0+(1-\beta_n)u_n-y_n\| \\
&= \|\beta_n(y_0-y_n)+(1-\beta_n)(u_n-y_n)\| \\
&= \|\beta_n(y_0-y_n)+(1-\beta_n)(\alpha_n y_n+(1-\alpha_n)(TJ_\lambda^{B_2}(y_n-\gamma B^*(Ax_n-By_n))-y_n)\| \\
&= \|\beta_n(y_0-y_n)+(1-\beta_n)(1-\alpha_n)(TJ_\lambda^{B_2}(y_n-\gamma B^*(Ax_n-By_n))-y_n)\| \\
&\leq \beta_n\|y_0-p\|+\beta_n\|y_n-p\| \\
&\quad +(1-\beta_n)(1-\alpha_n)\|TJ_\lambda^{B_1}(x_n-\gamma B^*(Ax_n-By_n))-y_n\| \qquad (3.36)
\end{aligned}$$

$$\begin{aligned}
\Phi_{n+1}(x_n,y_n) &= \|x_{n+1}-x_n\|+\|y_{n+1}-y_n\| \\
&\leq \beta_n\|x_0-p\|+\beta_n\|x_n-p\| \\
&\quad +(1-\beta_n)(1-\alpha_n)\|SJ_\lambda^{B_1}(x_n-\gamma A^*(Ax_n-By_n))-x_n\| \\
&\quad +\beta_n\|y_0-p\|+\beta_n\|y_n-p\| \\
&\quad +(1-\beta_n)(1-\alpha_n)\|TJ_\lambda^{B_1}(x_n-\gamma B^*(Ax_n-By_n))-y_n\| \qquad (3.37)
\end{aligned}$$

From equations (3.29), (3.32) and (3.33), $\lim_{n\to\infty}\Phi_{n+1}(x_n,y_n)=0$.

By Lemma 2.2, we have $F(S)$, $F(J_\lambda^{B_1})$, $F(T)$ and $F(J_\lambda^{B_2})$ are closed and convex, and hence Γ is also closed and convex. Let $(\hat{p},\hat{q})=P_\Gamma(u,v)$. By characterization of the metric projection, we get

$$\langle (u,v)-(\hat{p},\hat{q}),(x,y)-(\hat{p},\hat{q})\rangle \leq 0., \quad \forall z\in\Gamma. \qquad (3.38)$$

Now, since $\{x_n,y_n\}$ is bounded in $H_1\times H_2$, there exists $(\hat{p},\hat{q})\in H_1\times H_2$ and a subsequence $\{x_{n_i},y_{n_i}\}$ of $\{x_n,y_n\}$ such that $(x_{n_i},y_{n_i})\rightharpoonup(\hat{p},\hat{q})$ and Since $(\hat{p},\hat{q})\in\Gamma$, we obtain $S\hat{p}=\{\hat{p}\}$, $T\hat{q}=\{\hat{q}\}$, $J_\lambda^{B_1}\hat{p}=\{\hat{p}\}$ and $J_\lambda^{B_2}\hat{q}=\{\hat{q}\}$.

$$\limsup_{n\to\infty}\langle (u,v)-(\hat{p},\hat{q}),(x_n,y_n)-(\hat{p},\hat{q})\rangle \leq 0. \qquad (3.39)$$

To show this, since $\{x_n,y_n\}$ is bounded in $H_1\times H_2$, there exists $(\hat{p},\hat{q})\in H_1\times H_2$ and a subsequence $\{x_{n_i},y_{n_i}\}$ of $\{x_n,y_n\}$ such that $(x_{n_i},y_{n_i})\rightharpoonup(\hat{p},\hat{q})$ and

$$\begin{aligned}
\limsup_{n\to\infty}[\langle u &- \hat{p},x_n-\hat{p}\rangle+\langle v-\hat{q},y_n-\hat{q}\rangle] \\
&= \limsup_{n\to\infty}\langle (u,v)-(\hat{p},\hat{q}),(x_n,y_n)-(\hat{p},\hat{q})\rangle \\
&= \lim_{i\to\infty}\langle (u,v)-(\hat{p},\hat{q}),(x_{n_i},y_{n_i})-(\hat{p},\hat{q})\rangle. \qquad (3.40)
\end{aligned}$$

But $(x_{n_i},y_{n_i})\rightharpoonup(\hat{p},\hat{q})$ implies that $x_{n_i}\rightharpoonup\hat{p}$ and $y_{n_i}\rightharpoonup\hat{q}$. Hence from equation (3.29), we have $x_{n_i}\rightharpoonup\hat{p}$ and $v_{n_i}\rightharpoonup\hat{q}$, respectively. Now, since $(I-S)$ and $(I-T)$ are demiclosed at zero, from Equation (3.32) and (3.33) we get $\hat{p}\in F(S)$ and $\hat{q}\in F(T)$.

Next, we show that $A\hat{p}=B\hat{a}$. Observe that

$$\begin{aligned}
\|A\hat{p} &- B\hat{q}\|^2=\|A\hat{p}-Ax_{n_i}+Ax_{n_i}-By_{n_i}+By_{n_i}-B\hat{q}\|^2 \\
&= \|(A\hat{p}-Ax_{n_i}+By_{n_i}-B\hat{q})+(Ax_{n_i}-By_{n_i})\|^2 \\
&\leq \|Ax_{n_i}-By_{n_i}\|^2+2\langle A\hat{p}-B\hat{q},A\hat{p}-Ax_{n_i}+By_{n_i}-B\hat{q}\rangle
\end{aligned}$$

where the inequality follows from inequality (1.9). Since $x_{n_i} \rightharpoonup \hat{p}$ and $y_{n_i} \rightharpoonup \hat{q}$ as $i \to \infty$, it follows that $Ax_{n_i} \rightharpoonup A\hat{p}$ and $By_{n_i} \rightharpoonup B\hat{q}$ as $i \to \infty$ Taking limits on both sides, and making use of Equation (3.31), we get

$$\begin{aligned} \|A\hat{p} &- B\hat{q}\|^2 \leq \limsup_{i\to\infty} \|Ax_{n_i} - By_{n_i}\|^2 \\ &+2\limsup_{i\to\infty} \langle A\hat{p} - B\hat{q}, A\hat{p} - Ax_{n_i} + By_{n_i} - B\hat{q}\rangle \\ &= 0. \end{aligned} \tag{3.41}$$

The inequality (3.41) implies that $A\hat{p} = B\hat{q}$, that is $(\hat{p}, \hat{q}) \in \Gamma$.

Since $x_{n_i} \rightharpoonup \hat{p}$, $y_{n_i} \rightharpoonup \hat{q}$, $\|SJ_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n)) - x_n\| \to 0$ and $\|TJ_\lambda^{B_1 2}(y_n - \gamma B^*(Ax_n - By_n)) - y_n\| \to 0$ as $n \to \infty$, we have $x_{n_i} \rightharpoonup \hat{p}$ and $y_{n_i} \rightharpoonup \hat{q}$. By the demiclosedness of $I-S$ and $I-J_\lambda^{B_1}$ at zero, then $I-SJ_\lambda^{B_2}$ is also demiclosed at zero. Again by the demiclosedness of $I-T$ and $I-J_\lambda^{B_2}$ at zero, then $I-TJ_\lambda^{B_2}$ is also demiclosed at zero, and from equations (3.32) and (3.33), we get $\hat{p} \in F(SJ_\lambda^{B_1}) = F(S)\cap B_1^{-1}0$ and $\hat{q} \in F(TJ_\lambda^{B_2}) = F(T)\cap B_2^{-1}0$.

Now let us show that $\hat{p} \in B_1^{-1}0$. Let $\omega_n = J_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n)$, then we can easily prove that

$$\frac{1}{\lambda}\left(x_n - \omega_n - \gamma A^*(Ax_n - By_n)\right) \in B_1\omega_n$$

By the monotonicity of B_1, we have

$$\left\langle \omega_n - v, \frac{1}{\lambda}\left(x_n - \omega_n - \gamma A^*(Ax_n - By_n)\right) - w \right\rangle$$

for all $(v, w) \in G(B_1)$. Thus, we also have

$$\left\langle \omega_{n_i} - v, \frac{1}{\lambda}\left(x_{n_i} - \omega_{n_i} - \gamma A^*(Ax_{n_i} - By_{n_i})\right) - w \right\rangle \tag{3.42}$$

for all $(v, w) \in G(B_1)$. Since $\omega_{n_i} \rightharpoonup \hat{p}$, $\|\omega_{n_i} - J_\lambda^{B_1}(x_{n_i} - \gamma A^*(Ax_{n_i} - By_{n_i}))\| \to 0$. $Ax_{n_i} - By_{n_i} \to 0$ as $i \to \infty$, then by taking the limit as $i \to \infty$ in equation (3.42) yields

$$\langle \hat{p} - v, -w\rangle \leq 0$$

for all $(v, w) \in G(B_1)$. By the maximal monotonicity of B_1, we get $0 \in B_1\hat{p}$, that is, $\hat{p} \in B_1^{-1}0$.

Similarly for $\hat{q} \in B_2^{-1}0$. By the maximal monotonicity of B_2, we get $0 \in B_2\hat{q}$, that is, $\hat{q} \in B_2^{-1}0$. Hence, the sequence $\{(x_n, y_n)\}$ generated by equation (3.1) converges strongly to $(p.q) \in \Gamma$ as $n \to \infty$. This completes the proof of the theorem.

Corollary 3.2. *Let H_1 and H_2 be real Hilbert spaces. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$be two bounded linear operators, and $S : H_1 \to H_1$ be Lipschitzian quasi-pseudocontractive self maps of H_1 and $T : H_2 \to H_2$ be Lipschitzian quasi-pseudocontractive self maps of H_2 such that $(I - S)$ and $(I - T)$ are demiclosed at zero. If the solution set of SEFPP is nonempty* *(that is, $\Gamma = \{(x,y) : x \in F(S), y \in F(T), Ax = By\} \neq \emptyset$). Suppose that x_0, $x_1 \in H_1$ and y_0, $y_1 \in H_2$ are chosen arbitrarily. Let $\{(x_n, y_n)\}$ be the iterative sequence generated by*

$$(3.43)\quad \begin{array}{rcl} x_{n+1} &=& \beta_n x_0 + (1-\beta_n)u_n \\ u_n &=& \alpha_n x_n + (1-\alpha_n)S(x_n - \gamma A^*(Ax_n - By_n)) \\ y_{n+1} &=& \beta_n y_0 + (1-\beta_n)v_n \\ v_n &=& \alpha_n y_n + (1-\alpha_n)T(y_n + \gamma B^*(Ax_n - By_n)) \end{array} \quad n \geq 1.$$

where the parameter γ and the sequences $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$ satisfy the conditions: (i) $\gamma \in \left(0, \frac{2(1-\beta_n L^2)+\beta_n L^2(\|A\|^2+\|B\|^2)}{(1+2(L+1)^2)(\|A\|^2+\|B\|^2)}\right)$, (ii) $\sum_{n=1}^{\infty} \alpha_n < \infty$, (iii) $\lim_{n\to\infty} \beta_n = 0$ and (iv) $\sum_{n=1}^{\infty} \beta_n = \infty$. Then,

(a) $\lim_{n\to\infty} \Phi_n(p,q)$ *exists for each* $(p,q) \in \Gamma$,

(b) $\lim_{n\to\infty} \|x_n - S(x_n - \gamma A^*(Ax_n - By_n))\| = \lim_{n\to\infty} \|y_n - T(y_n + \gamma B^*(Ax_n - By_n))\| = 0$,

(c) $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $(p,q) \in \Gamma$.

Corollary 3.3. *Let H_1 and H_2 be real Hilbert spaces. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators and if the solution set of split Equality Null Point Problem (SENPP) is nonempty (that is, $\Gamma = \{(x,y) : x \in B_1^{-1}0, y \in B_2^{-1}0, Ax = By\} \neq \emptyset$) where $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are two set valued maximal monotone mappings. Suppose that x_0, $x_1 \in H_1$ and y_0, $y_1 \in H_2$ are chosen arbitrarily. Let $\{(x_n, y_n)\}$ be the iterative sequence generated by*

$$(3.44)\quad \begin{array}{rcl} x_{n+1} &=& \beta_n x_0 + (1-\beta_n)u_n \\ u_n &=& \alpha_n x_n + (1-\alpha_n)J_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n)) \\ y_{n+1} &=& \beta_n y_0 + (1-\beta_n)v_n \\ v_n &=& \alpha_n y_n + (1-\alpha_n)J_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n)) \end{array} \quad n \geq 1.$$

where the parameter γ and the sequences $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$ satisfy the conditions: (i) $\gamma \in \left(0, \frac{2(1-\beta_n L^2)+\beta_n L^2(\|A\|^2+\|B\|^2)}{(1+2(L+1)^2)(\|A\|^2+\|B\|^2)}\right)$, (ii) $\sum_{n=1}^{\infty} \alpha_n < \infty$, (iii) $\lim_{n\to\infty} \beta_n = 0$ and (iv) $\sum_{n=1}^{\infty} \beta_n = \infty$. Then,

(a) $\lim_{n\to\infty} \Phi_n(p,q)$ *exists for each* $(p,q) \in \Gamma$,

(b) $\lim_{n\to\infty} \|x_n - J_\lambda^{B_1}(x_n - \gamma A^*(Ax_n - By_n))\| = \lim_{n\to\infty} \|y_n - J_\lambda^{B_2}(y_n + \gamma B^*(Ax_n - By_n))\| = 0$,

(c) $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $(p,q) \in \Gamma$.

4. Applications

Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem is to find $\bar{x} \in C$ such that $f(\bar{x}, y) \geq 0$ for all $y \in C$. The set of such solutions is denoted by $EP(f)$. Numerous problems in physics, optimization, and economics reduce to finding a solution to the equilibrium problem (see [5]).

Lemma 4.1. *For solving the equilibrium problem, they assumed that the bifunction f satisfies the following conditions:*

- *(A1)* $f(x,x)=0$ *for all* $x \in C$,
- *(A2)* f *is monotone, that is,* $f(x,y)+f(y,x) \leq 0$ *for all* $x,y \in C$,
- *(A3) for every* $x,y,z \in C$, $\limsup_{t\downarrow 0} f(tz+(1-t)x,y) \leq f(x,y)$,
- *(A4)* $f(x,\cdot)$ *is convex and lower semicontinuous for each* $x \in C$.

Lemma 4.2. [5]. *Let C be a nonempty closed convex subset of H, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1) – (A4). If $r>0$ and $x \in H$, then there exists $z \in C$ such that*

$$f(z,y)+\frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall y \in C. \tag{4.1}$$

Lemma 4.3. [17]. *Let C be a nonempty closed convex subset of H, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1) – (A4). For $r>0$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x)=\{z \in C : f(z,y)+\frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall y \in C\}. \tag{4.2}$$

Then the following hold:

- *(i)* T_r *is single valued,*
- *(ii)* T_r *is firmly nonexpansive, that is, for any* $x,y \in H$

$$\langle x-y, T_rx-T_ry\rangle \geq \|T_rx-T_ry\|^2, \tag{4.3}$$

- *(iii)* $Fix(T_r)=EP(f)$,
- *(iv)* $EP(f)$ *is closed and convex.*

Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $f_1 : C \times C \to \mathbb{R}$ and $f_2 : Q \times Q \to \mathbb{R}$ be two bifunctions and $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators, then the split equality equilibrium problem (SEEP) is to find a point $(x^*,y^*) \in C \times Q$ such that

$$\left.\begin{array}{l} f_1(x^*,x) \geq 0 \ \forall x \in C \ \text{ and} \\ f_2(y^*,y) \geq 0 \ \forall y \in Q \end{array}\right\} \tag{4.4}$$

Then above problem is to find a point $(x^*,y^*) \in C \times Q$ such that

$$x^* \in EP(f_1) \ \text{ and } \ y^* \in EP(f_2) : Ax^*=By^*. \tag{4.5}$$

Lemma 4.4. [25] *Let C be a nonempty closed convex subset of H, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1) – (A4). Define A_f as follows:*

$$A_f(x)=\begin{cases} \{z \in H : f(z,y) \geq \langle y-x, z\rangle, \ \forall y \in C\}, & \textit{if } x \in C \\ \emptyset & x \textit{ if } \notin C \end{cases} \tag{4.6}$$

Then the following hold: (i) A_f is msximal monotone, (ii) $EP(f)=A_f^{-1}0$, (ii) $T_r^f=(I+rA_f)^{-1}0, \ r>0$.

Let H be a real Hilbert space, and let f be a proper lower semicontinuous convex function of H into $(-\infty, +\infty]$. Then the subdifferential ∂f of f is defined as

$$\partial f(x) = \{z \in H : f(y) - f(x) \geq \langle z, y - x\rangle, \ \ \forall y \in H\} \tag{4.7}$$

for all $x \in H$. [24] claimed that ∂f is a maximal monotone operator. Let C be a nonempty closed convex subset of H, and let δ_C be the indicator function of C. That is,

$$\delta_C(x) = \begin{cases} 0 & x \in C \\ +\infty & x \notin C \end{cases} \tag{4.8}$$

Since δ_C is a proper lower semicontinuous convex function on H, the subdifferential ∂_{δ_C} of δ_C is a maximal monotone operator. The resolvent J_λ of ∂_{δ_C} for $\lambda > 0$ is defined by

$$J_\lambda x = (I + \lambda \partial_{\delta_C})^{-1} x, \quad \forall x \in H. \tag{4.9}$$

we have

$$\begin{aligned} u = (I + \lambda \partial_{\delta_C})^{-1} x \quad &\Leftrightarrow \quad x \in u + \lambda \partial_{\delta_C} u \\ &\Leftrightarrow \quad x \in u + \lambda N_C u \Leftrightarrow x - u \in \lambda N_C u \\ &\Leftrightarrow \quad \frac{1}{\lambda}\langle x - u, y - u\rangle \leq 0, \ \ \forall y \in C \\ &\Leftrightarrow \quad u = P_C x \end{aligned} \tag{4.10}$$

where $N_C u = \{z \in H : \langle z, x - u\rangle \leq 0 \ \forall y \in C\}$. The variational inequality problem for nonlinear operator A is to find $z \in C$ such that

$$\langle Az, y - z\rangle \geq 0 \ \ \forall y \in C. \tag{4.11}$$

The set of its solutions is denoted by $VI(C, A)$. Then we have

$$\begin{aligned} z \quad &\in \quad VI(C, A) \\ &\Leftrightarrow \quad \langle Az, x - z\rangle \geq 0 \ \ \forall y \in C \\ &\Leftrightarrow \quad \langle -Az, x - z\rangle \leq 0 \ \ \forall y \in C \\ &\Leftrightarrow \quad -Az \in N_C z \\ &\Leftrightarrow \quad 0 \in Az + N_C z \Leftrightarrow 0 \in Az + \partial_{\delta_C} z \\ &\Leftrightarrow \quad z \in (A + \partial_{\delta_C})^{-1} 0. \end{aligned} \tag{4.12}$$

With (4.12), we can obtain the strong convergence theorem for the variational inequality problem.

Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be two quasi pseudocontractiv mappings and $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators.

The Split Equality Variational Inequality Problem denoted by SEVIP is to find a point $(u^*, v^*) \in C \times Q$ such that

$$\left.\begin{array}{l}\langle u - u^*, SJ_\lambda^{B_1}(u^* - \gamma A^*(Ax_n - By_n))\rangle \geq 0 \ \ \forall u \in C \ \text{ and} \\ \langle v - v^*, TJ_\lambda^{B_1}(v^* + \gamma B^*(Ax_n - By_n))\rangle \geq 0 \ \ \forall v \in Q. \\ \text{such that } \ Ax_n = By_n.\end{array}\right\} \tag{4.13}$$

Let D be the solution set of the SEVIP given by

$$D = \{u^* \in VI(C,S), v^* \in VI(Q,T) : Ax_n = By_n\} \tag{4.14}$$

We observe that $u^*, v^* \in SEVIP$ if and only if $u^* = SJ_\lambda^{B_1}(u^* - \gamma A^*(Ax_n - By_n))$ and $v^* = SJ_\lambda^{B_1}(v^* + \gamma B^*(Ax_n - By_n))$.

Let $f : H \to (-\infty, +\infty]$ be a function, we define the set of minimizer of f by

$$\text{Argmin} f := \{x \in H : f(x) \leq f(z), \forall z \in H\}. \tag{4.15}$$

If f is a proper, lower semicontinuous and convex function, then ∂f is a maximal monotone operator. Moreover,

$$x \in (\partial f)^{-1}0 \Longleftrightarrow 0 \in \partial f(x) \Leftrightarrow f(x) \leq f(z), \forall z \in H \Leftrightarrow x \in \text{Argmin} f, \tag{4.16}$$

that is, Argmin $f = (\partial f)^{-1}0$. In this case, the resolvent of ∂f is called the proximity operator of f.

Let H_1 and H_2 be real Hilbert spaces. Let $f : H_1 \to (-\infty, +\infty]$ and $g : H_2 \to (-\infty, +\infty]$ be proper, lower semicontinuous and convex functions. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators, the Split Equality Optimization Problem (SEOP) is the problem of finding $(x^*, y^*) \in H_1 \times H_2$ such that

$$x^* \in \text{Argmin } f \ \text{ and } \ y^* \in \text{Argmin } g, \ \text{ such that } \ Ax^* = By^*. \tag{4.17}$$

Denote by $\partial f = B_1$ and $\partial g = B_2$. Since x^* and y^* are the minimum of f on H_1 and g on H_2, respectively for any $\lambda > 0$, we have

$$\begin{aligned} x^* &= F(S) \cap (\partial f)^{-1}0 = Fix(SJ_\lambda^{\partial f}) \ \text{ and} \\ y^* &= F(T) \cap (\partial g)^{-1}0 = Fix(TJ_\lambda^{\partial g}). \end{aligned} \tag{4.18}$$

This implies that the split euqlity optimization problem (4.17) is equivalent to the split common fixed point and null point problem SEFPNPP.

4.1. **Split Equality feasibility Problem (SEFP).**

Theorem 4.5. *Let H_1 and H_2 be real Hilbert spaces and C and Q be nonempty closed convex subsets of H_1 and H_2 respectively. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators, and $S : H_1 \to H_1$ be Lipschitzian quasi-pseudocontractive self maps of H_1 and $T : H_2 \to H_2$ be Lipschitzian quasi-pseudocontractive self maps of H_2 such that $(I - S)$ and $(I - T)$ are demiclosed at zero. If the solution set of SEFP is nonempty (that is, $\Gamma = \{(x, y) : x \in F(S) \cap C \ , y \in F(T) \cap Q : Ax = By\} \neq \emptyset$). Suppose* *that* x_0, $x_1 \in H_1$ *and* y_0, $y_1 \in H_2$ *are chosen arbitrarily. Let* $\{(x_n, y_n)\}$ *be the iterative sequence generated by*

$$(4.19)\quad \begin{array}{rcl} x_{n+1} &=& \beta_n x_0 + (1-\beta_n)u_n \\ u_n &=& \alpha_n x_n + (1-\alpha_n)SP_C(x_n - \gamma A^*(Ax_n - By_n)) \\ y_{n+1} &=& \beta_n y_0 + (1-\beta_n)v_n \\ v_n &=& \alpha_n y_n + (1-\alpha_n)TP_Q(y_n + \gamma B^*(Ax_n - By_n)) \end{array} \quad n \geq 1$$

where the parameter γ *and the sequences* $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$ *satisfy the conditions: (i)* $\gamma \in \left(0, \frac{2(1-\beta_n L^2)+\beta_n L^2(\|A\|^2+\|B\|^2)}{(1+2(L+1)^2)(\|A\|^2+\|B\|^2)}\right)$, *(ii)* $\sum_{n=1}^{\infty} \alpha_n < \infty$, *(iii)* $\lim_{n\to\infty} \beta_n = 0$ *and (iv)* $\sum_{n=1}^{\infty} \beta_n = \infty$. *Then,*

(a) $\lim_{n\to\infty} \Phi_n(p,q)$ *exists for each* $(p,q) \in \Gamma$,

(b) $\lim_{n\to\infty} \|x_n - SP_C(x_n - \gamma A^*(Ax_n - By_n))\| = \lim_{n\to\infty} \|y_n - TP_Q(y_n + \gamma B^*(Ax_n - By_n))\| = 0$,

(c) $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $(p,q) \in \Gamma$.

Proof: Set $B_1 := \partial\delta_C$ and $B_2 := \partial\delta_Q$. Then B_1 and B_2 are maximal monotone such that $J_\lambda^{B_1} = P_C$ and $J_\lambda^{B_2} = P_Q$ for $\lambda > 0$. We also have $B_1^{-1}0 = C$ and $B_2^{-1}0 = Q$. Hence the result is obtained directly by Theorem 3.1.

4.2. **Split Equality Equilibrium Problem (SEEP).**

Theorem 4.6. *Let* C *and* Q *be nonempty closed convex subsets of* H_1 *and* H_2, *respectively.* $f_1 : C \times C \to \mathbb{R}$ *and* $f_2 : C \times C \to \mathbb{R}$ *be bifunctions satisfying (A1) – (A4) and let* $T_{r_1}^{f_1}$ *and* $T_{r_2}^{f_2}$ *be resolvents of* A_{f_1} *and* A_{f_2} *in Lemma* **??**, *respectively for* r_1, $r_2 > 0$. *Let* $A : H_1 \to H_3$ *and* $B : H_2 \to H_3$ *be bounded linear operators, and* $S : H_1 \to H_1$ *be Lipschitzian quasi-pseudocontractive self maps of* H_1 *and* $T : H_2 \to H_2$ *be Lipschitzian quasi-pseudocontractive self maps of* H_2 *such that* $(I-S)$ *and* $(I-T)$ *are demiclosed at zero. If the solution set of SEEP (4.5) is nonempty (that is,* $\Gamma = \{(x,y) : x \in F(S) \cap EP(f_1), y \in F(T) \cap EP(f_2), : Ax = By\} \neq \emptyset$*). Suppose that* x_0, $x_1 \in H_1$ *and* y_0, $y_1 \in H_2$ *are chosen arbitrarily. Let* $\{(x_n, y_n)\}$ *be the iterative sequence generated by*

$$(4.20)\quad \begin{array}{rcl} x_{n+1} &=& \beta_n x_0 + (1-\beta_n)u_n \\ u_n &=& \alpha_n x_n + (1-\alpha_n)ST_{r_1}^{f_1}(x_n - \gamma A^*(Ax_n - By_n)) \\ y_{n+1} &=& \beta_n y_0 + (1-\beta_n)v_n \\ v_n &=& \alpha_n y_n + (1-\alpha_n)TT_{r_2}^{f_2}(y_n + \gamma B^*(Ax_n - By_n)) \end{array} \quad n \geq 1$$

where the parameter γ *and the sequences* $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$ *satisfy the conditions: (i)* $\gamma \in \left(0, \frac{2(1-\beta_n L^2)+\beta_n L^2(\|A\|^2+\|B\|^2)}{(1+2(L+1)^2)(\|A\|^2+\|B\|^2)}\right)$, *(ii)* $\sum_{n=1}^{\infty} \alpha_n < \infty$, *(iii)* $\lim_{n\to\infty} \beta_n = 0$ *and (iv)* $\sum_{n=1}^{\infty} \beta_n = \infty$. *Then,*

(a) $\lim_{n\to\infty} \Phi_n(p,q)$ *exists for each* $(p,q) \in \Gamma$,

(b) $\lim_{n\to\infty} \|x_n - ST_{r_1}^{f_1}(x_n - \gamma A^*(Ax_n - By_n))\| = \lim_{n\to\infty} \|y_n - TT_{r_2}^{f_2}(y_n + \gamma B^*(Ax_n - By_n))\| = 0$,

(c) $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $(p, q) \in \Gamma$.

Proof : We set $B_1 := A_{f_1}$ and $B_2 := A_{f_2}$. By Lemma 4.4, we know that B_1 and B_2 are maximal monotone, $EP(f_1) = B_1^{-1}0,\ EP(f_2) = B_2^{-1}0,\ T_{r_1}^{f_1} = J_\lambda^{B_1}$ and $T_{r_2}^{f_2} = J_\lambda^{B_2}$, so the result is obtained directly by Theorem 3.1.

4.3. **Split Equality Variational inequality Problem (SEVIP).**

Theorem 4.7. *Let* H_1 *and* H_2 *be Hilbert spaces,* $A : H_1 \to H_3$ *and* $B : H_2 \to H_3$ *be bounded linear operators, and* $S : H_1 \to H_1$ *be Lipschitzian quasi-pseudocontractive self maps of* H_1 *and* $T : H_2 \to H_2$ *be Lipschitzian quasi-pseudocontractive self maps of* H_2 *such that* $(I - S)$ *and* $(I - T)$ *are demiclosed at zero. Let* A^* *denotes the adjoint of* A*. Let* $B_1 : H_1 \to 2^{H_1}$ *and* $B_2 : H_2 \to 2^{H_2}$ *be two set valued maximal monotone mappings and* $\gamma,\ \lambda > 0$*. Given any* $x^* \in H_1$*,* $y^* \in H_2$

(i) *if* x^* *and* y^* *are solutions of SEVIP, then* $SJ_\lambda^{B_1}(x^* - \gamma A^*(Ax^* - By^*)) = x^*$ *and* $TJ_\lambda^{B_2}(y_n + \gamma B^*(Ax^* - By^*)) = y^*$*,*

(ii) *Suppose that* $SJ_\lambda^{B_1}(x^* - \gamma A^*(Ax^* - By^*)) = x^*$ *and* $TJ_\lambda^{B_2}(y^* + \gamma B^*(Ax^* - By^*)) = y^*$*, and the solution set of SEVIP are not empty, then* x^* *and* y^* *are solutions of SEVIP.*

Proof : (i) Suppose that $x^* \in H_1$ is a solution of SEVIP, then $x^* \in F(S) \cap B_1^{-1}0$ and $y^* \in F(T) \cap B_2^{-1}0$. It is can be seen that $SJ_\lambda^{B_1}(x^* - \gamma A^*(Ax_n - By_n)) = x^*$ and $TJ_\lambda^{B_2}(y^* + \gamma B^*(Ax^* - By^*)) = y^*$.
(ii) Suppose that w^*, ϖ^* is the solution of SEVIP and $SJ_\lambda^{B_1}(x^* - \gamma A^*(Ax^* - By^*)) = x^*$ and $TJ_\lambda^{B_2}(y^* + \gamma B^*(Ax^* - By^*)) = y^*$,

$$\langle x^* - \gamma A^*(Ax^* - By^*) - x^*, x^* - w^*\rangle + \langle y^* + \gamma A^*(Ax^* - By^*) - y^*, y^* - \varpi^*\rangle \geq 0$$

for each $w^* \in F(S) \cap B_1^{-1}0$, that is,

$$\langle A^*(Ax^* - By^*), x^* - w^*\rangle \leq 0$$

for each $\varpi^* \in F(T) \cap B_2^{-1}0$,

$$\langle B^*(Ax^* - By^*), y^* - \varpi^*\rangle \leq 0$$

w^*, ϖ^* is the solution of SEVIP.

4.4. **Split Equality Optimization Problem (SEOP).**

Theorem 4.8. *Let* H_1 *and* H_2 *be Hilbert spaces. Let* $f : H_1 \to \mathbb{R}$ *and* $g : H_2 \to \mathbb{R}$ *be proper lower semicontinuous convex function of* H *into* $(-\infty, +\infty]$*. Let* $A : H_1 \to H_3$ *and* $B : H_2 \to H_3$ *be bounded linear operators, and* $S : H_1 \to H_1$ *be Lipschitzian quasi-pseudocontractive self maps of* H_1 *and* $T : H_2 \to H_2$ *be Lipschitzian quasi-pseudocontractive self maps of* H_2 *such that* $(I - S)$ *and* $(I - T)$ *are demiclosed at zero. If the solution set of SEOP (4.17) is nonempty (that is,* $\Gamma = \{(x, y) : x \in F(S) \cap (\partial f)^{-1}0, y \in$

$F(T)\cap(\partial g)^{-1}0 : Ax = By\} \neq \emptyset)$. Suppose that x_0, $x_1 \in H_1$ and y_0, $y_1 \in H_2$ are chosen arbitrarily. Let $\{(x_n, y_n)\}$ be the iterative sequence generated by

$$
\begin{array}{rcl}
x_{n+1} & = & \beta_n x_0 + (1-\beta_n)u_n \\
u_n & = & \alpha_n x_n + (1-\alpha_n)SJ_\lambda^{\partial f}(x_n - \gamma A^*(Ax_n - By_n)) \\
y_{n+1} & = & \beta_n y_0 + (1-\beta_n)v_n \\
v_n & = & \alpha_n y_n + (1-\alpha_n)TJ_\lambda^{\partial f}(y_n + \gamma B^*(Ax_n - By_n))
\end{array}
\quad n \geq 1 \tag{4.21}
$$

where the parameter γ and the sequences $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$ satisfy the conditions: (i) $\gamma \in \left(0, \frac{2(1-\beta_n L^2)+\beta_n L^2(\|A\|^2+\|B\|^2)}{(1+2(L+1)^2)(\|A\|^2+\|B\|^2)}\right)$, (ii) $\sum_{n=1}^\infty \alpha_n < \infty$, (iii) $\lim_{n\to\infty} \beta_n = 0$ and (iv) $\sum_{n=1}^\infty \beta_n = \infty$. Then,

(a) $\lim_{n\to\infty} \Phi_n(p,q)$ *exists for each* $(p,q) \in \Gamma$,

(b) $\lim_{n\to\infty} \|x_n - SJ_\lambda^{\partial f}(x_n - \gamma A^*(Ax_n - By_n))\| = \lim_{n\to\infty} \|y_n - TJ_\lambda^{\partial g}(y_n + \gamma B^*(Ax_n - By_n))\| = 0$,

(c) $\{x_n\}_{n=1}^\infty$ *converges strongly to* $(p,q) \in \Gamma$.

Proof: Set $B_1 := \partial f$ and $B_2 := \partial g$. Hence the result is obtained directly by Theorem 3.1.

5. Conclusion

In this paper:

(1) We obtained strong convergence results from our algorithm without imposing compactness type condition (demi-compactness) on the mapping S and T which appear to be a stronger condition.
(2) Chang et al. [14] showed that strong convergence is guaranteed if the maps S and T are semi-compact whereas the condition is not required in our theorem.
(3) The efficiency and implementation of iterative algorithm does not require the calculation or estimation of the operator norms $\|A\|$ and $\|B\|$ which may at times be as difficult as solving the original problem itself.
(4) Our work unify the split equality fixed point problem, split equality null point problem and other related fixed point problems.
(5) The above results for quasi-pseudocontractive maps are also valid for firmly quasi nonexpansive, quasi nonexpansive maps, demicontractive mappings and hence our results improve and extend many results in the literature [14, 16, 21].

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