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Research Paper

OPTIMALITY CONDITIONS FOR SOLVING NONCONVEX SET-VALUED EQUILIBRIUM PROBLEMS

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Abstract. In this paper, sufficient conditions ensuring the existence of solutions for set-valued equilibrium problems are obtained. The convexity assumption on the whole domain is not necessary and just the closure of a quasi-self-segment-dense subset of the domain is convex. Using a KKM theorem and a notion of Q-selected preserving $\mathbb{R}^*_-$-intersection ($\mathbb{R}^*_-$-inclusion) for set-valued mapping, existence results are established in real Hausdorff topological vector spaces.

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1. **Introduction**

Equilibrium problem theory can be viewed as a significant area of nonlinear analysis, where the focus is on applications to optimization, variational inequalities, fixed point theory, and etc.. The reader can find more discussions about various aspects of equilibrium problems and its applications in[1, 2, 3, 4, 5] and the references therein .

In 2016, results for the existence of solutions of set-valued equilibrium problems were established by László and Viorel [6]. The assumptions were considered just on a self-segment-dense subset of the domain of involved set-valued bifunction. Afterward, a notion of *locally segment-dense sets* presented to obtain existence results for single-valued equilibrium problems under assumptions imposed on a locally segment-dense subset of the domain [7].

Quite recently, quasi-self-segment-dense subsets of the domain of involved bifunctions were introduced [8]. The class of quasi-self-segment-dense sets properly includes the class of locally segment-dense as well as self-segment-dense sets. The main disadvantage of the latter notion is that the considered domain is not necessarily convex, and just the closure of a quasi-self-segment-dense subset of the domain should be convex.

In this paper, new optimality conditions for the existence of solutions of nonconvex set-valued equilibrium problems are established. No closedness assumptions or convexity structures on the whole domain are considered and the existence results are proved via a KKM theorem in the setting of Hausdorff topological vector spaces.

The paper is organized as follows. In Section 2, definitions and auxiliary tools required for proofs of results in the next sections are recalled. The notion of a quasi-self-segment-dense set is also presented and a notion of Q-selected preserving $\mathbb{R}^*_-$-intersection ($\mathbb{R}^*_-$-inclusion) for

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set-valued mappings is introduced. In section 3, sufficient conditions ensuring the existence of solutions for set-valued equilibrium problems with domains which are neither convex nor closed, are provided.

2. Preliminaries

In this paper, let X be a Hausdorff topological vector space, and let K be a nonempty convex subset of X. For given elements $x, y \in K$, $[x, y] := \{(1-t)x + ty : t \in [0,1]\}$ is the closed line segment. The semi-open segments $[x, y[,]x, y]$ and the open segment $]x, y[$ are defined analogously. If A is a nonempty subset of X, cl$A$ and conv$A$ denote the closure and the convex hull of A, respectively.

In the sequel, the notations $\mathbb{R}_+ = [0, +\infty[$, $\mathbb{R}_+^* =]0, +\infty[$, $\mathbb{R}_- =]-\infty, 0]$, and $\mathbb{R}_-^* =]-\infty, 0[$, where $\mathbb{R} =]-\infty, +\infty[$ is the set of real numbers, are used.

Let $K \subseteq X$ be nonempty, and let $f : K \times K \to \mathbb{R}$ be a bifunction. The single-valued equilibrium problem is to find a point $\bar{x} \in K$ such that

$$f(\bar{x}, y) \geq 0, \quad \forall y \in K \quad \text{(EP)}.$$

To formulate a set-valued equilibrium problem, let $F : K \times K \rightrightarrows \mathbb{R}$ be a set-valued mapping. The goal of the set-valued equilibrium problem is to find $\bar{x} \in K$ such that

$$F(\bar{x}, y) \subseteq \mathbb{R}_+, \quad \forall y \in K \quad \text{(SEP)}.$$

It is also useful to consider the weak set-valued equilibrium problem which is to find $\bar{x} \in K$ such that

$$F(\bar{x}, y) \cap \mathbb{R}_+ \neq \emptyset, \quad \forall y \in K \quad \text{(WSEP)}.$$

$S(F, K)$ (resp. $S_W(F, K)$) denotes the set of all solutions of SEP (resp. WSEP).

In what follows, some notions of convexity for set-valued mappings are presented.

Definition 2.1 ([9])**.** Let K be a convex subset of X, and let $G : K \rightrightarrows \mathbb{R}$ be a set-valued mapping. We say that G is

- lower convex, if for every $x_1, x_2, \ldots, x_n \in K$ and $t_1, t_2, \ldots, t_n \geq 0$ such that $\sum_{i=1}^n t_i = 1$, it holds that
$$\sum_{i=1}^n t_i G(x_i) \subseteq G(\sum_{i=1}^n t_i x_i) + \mathbb{R}_+;$$
- lower quasiconvex, if for every $x_1, x_2, \ldots, x_n \in K$ and $t_1, t_2, \ldots, t_n \geq 0$ satisfying $\sum_{i=1}^n t_i = 1$, then
$$\bigcap_{i=1}^n (G(x_i) + \mathbb{R}_+) \subseteq G(\sum_{i=1}^n t_i x_i) + \mathbb{R}_+;$$
- upper convex, if for every $x_1, x_2, \ldots, x_n \in K$ and $t_1, t_2, \ldots, t_n \geq 0$ satisfying $\sum_{i=1}^n t_i = 1$, then
$$G(\sum_{i=1}^n t_i x_i) \subseteq (\sum_{i=1}^n t_i G(x_i)) - \mathbb{R}_+.$$

It is easy to verify that every lower convex set-valued mapping is lower quasiconvex. In the following, the definition of the upper quasiconvexity for set-valued mappings is presented.

Definition 2.2 ([10]). Let K be a convex subset of X, and let $G : K \rightrightarrows \mathbb{R}$ be a set-valued mapping. We say that G is upper quasiconvex, if for every $x_1, x_2, \ldots, x_n \in K$ and $t_1, t_2, \ldots, t_n \geq 0$ such that $\sum_{i=1}^{n} t_i = 1$, it holds that

$$G(\sum_{i=1}^{n} t_i x_i) - \mathbb{R}_+ \subseteq \bigcup_{i=1}^{n} (G(x_i) - \mathbb{R}_+).$$

Proposition 2.3 ([10]). *Every upper convex set-valued mapping $G : K \to \mathbb{R}$ is upper quasiconvex.*

One can easily provide examples of a set-valued mapping which is upper quasiconvex but not upper convex (see [10]).

Notice that $G : K \rightrightarrows \mathbb{R}$ defined by $G(x) := \{g(x)\}$, where $g : K \rightrightarrows \mathbb{R}$ is a real single-valued mapping, then the notion of upper (lower) convex mapping of G is equivalent to the classical convexity of g.

Recall that the set-valued mapping $G : X \rightrightarrows \mathbb{R}$ is lower semicontinuous at a point $x_0 \in K$ iff for every open subset V of $\mathbb{R}$ such that $G(x_0) \cap V \neq \emptyset$, there exists a neighborhood $U \subseteq X$ of x_0 such that, for every point $y \in U \cap K$ one has $G(y) \cap V \neq \emptyset$. Similarly, the set-valued mapping $G : K \rightrightarrows \mathbb{R}$ is called upper semicontinuous at a point $x_0 \in K$ iff for every open subset V of $\mathbb{R}$ such that $G(x_0) \subseteq V$, there exists a neighborhood $U \subseteq X$ of x_0 such that, one has $G(U) \subseteq V$.

The set-valued mapping G is lower semicontinuous on K iff it is lower semicontinuous at every point of K. The upper semicontinuity on K are analogously defined. We say that a set-valued mapping $G : X \rightrightarrows Y$ is lower semicontinuous (resp. upper semicontinuous) on a subset S of K if it is lower semicontinuous (resp. upper semicontinuous) at every point of S.

Next, some generalizations of upper and lower semicontinuity for set-valued mappins are presented.

Definition 2.4 ([10]). Let K be a nonempty subset of X, and let S and T be two subsets of K. Suppose that $F : K \times K \rightrightarrows \mathbb{R}$ is a set-valued bifunction and $\lambda \in \mathbb{R}$. F is called λ-transfer upper semicontinuous on $S \times T$ if for every $x \in S$ and $y \in T$, $F(x, y) \subseteq]-\infty, \lambda[$ implies that there exist $y' \in T$ and a neighborhood U of x such that $F(z, y') \subseteq]-\infty, \lambda[$ for all $z \in U \cap S$. F is called λ-transfer lower semicontinuous on $S \times T$ if for every $x \in S$ and $y \in T$, $F(x, y) \cap]-\infty, \lambda[\neq \emptyset$ implies that there exist $y' \in T$ and a neighborhood U of x such that $F(z, y') \cap]-\infty, \lambda[\neq \emptyset$ for all $z \in U \cap S$.

It is easy to verify that if for every $y \in T$, $F(\cdot, y)$ is upper (resp. lower) semicontinuous on S, then it is λ-transfer upper (resp. lower) semicontinuous for every $\lambda \in \mathbb{R}$ on $S \times T$ (take $y' = y$ in the latter definition).

Remark 2.5. It is worth mentioning that to establish the existence results, it is required for the bifunction F to be just 0-transfer upper (lower) semicontinuous on $K_0 \times Q$, where Q is a q-s-s-d subset of K and K_0 is a nonempty compact subset of K. Hence, the continuity conditions which we use are weaker than the ones in the literature.

2.1. **Quasi-self-segment-dense sets.** László and Viorel [6] introduced *self-segment-dense* subsets. Given a convex subset K of X. The set $D \subseteq K$ is called self-segment-dense in K if

(i) D is dense in K;
(ii) for every $x, y \in D, \mathrm{cl}([\mathrm{x}, \mathrm{y}] \cap \mathrm{D}) = [\mathrm{x}, \mathrm{y}]$.

It must be noted that the notion of a self-segment-dense set coincide to the notion of a dense set, in one dimension. Afterward, a concept of *locally segment-dense* sets was presented in [7].

Definition 2.6. Given a convex subset K of X. The set $D \subseteq K$ is said to be locally segment-dense in K, iff

(i) for every $x \in D$ and $y \in K$, the set $]x, y] \cap D$ is nonempty;
(ii) for every $x, y \in D, \mathrm{cl}([\mathrm{x}, \mathrm{y}] \cap \mathrm{D}) = [\mathrm{x}, \mathrm{y}]$.

To establish existence results for nonconvex set-valued equilibrium problems, the following notion, proposed in [8], is used.

Definition 2.7. Let X be a Hausdorff topological vector space, and let Q be a nonempty subset of X. We say that the set Q is quasi-self-segment-dense (in short, q-s-s-d) if for every $x, y \in Q, \mathrm{cl}([\mathrm{x}, \mathrm{y}] \cap \mathrm{Q}) = [\mathrm{x}, \mathrm{y}]$.

Obviously, if $Q \subseteq K$ is self-segment-dense or locally segment-dense, then Q is a q-s-s-d subset of K. In fact, the class of the q-s-s-d sets is essentially larger than that of self-segment-dense sets as well as locally segment-dense sets.

Unlike self-segment-dense sets which K is assumed to be convex, a set K which contains a q-s-s-d set Q is not necessarily convex. For example, let $X = \mathbb{R}, K = (]-\infty, -1] \cap \mathbb{Q}) \cup [0, 1[$ and $Q = [0, 1[\cap \mathbb{Q}$, then Q is a q-s-s-d subset of K. Notice that both Q and K are nonconvex. Note that Q is not dense in K and so Q is not self-segment-dense. It is easy to see that Q is not locally segment-dense, since for $x = 0 \in Q$ and $y = -1 \in K$, we have $]x, y] \cap Q = \emptyset$.

Remark 2.8. Let K be a nonempty subset of X. Then every convex subset of K is quasi-self-segment-dense. Hence one can provide many examples of quasi-segment-dense sets which are not dense in K and so they are not self-segment-dense in K. On the other hand, one can easily verify that q-s-s-d sets are self-segment-dense in their closures.

See [8, 10] for further explanations and examples of q-s-s-d sets.

The following key lemma to obtain existence results is used. Lemma 2.9 is a generalization of Lemma 3.1 in [6] to the case of real Hausdorff topological vector spaces.

Lemma 2.9 ([10]). *Let X be a real Hausdorff topological vector space, and let D be a subset of K satisfying*

$$\mathrm{cl}([\mathrm{x}, \mathrm{y}] \cap \mathrm{D}) = [\mathrm{x}, \mathrm{y}], \qquad \forall \mathrm{x}, \mathrm{y} \in \mathrm{D}.$$

Then for all finite subsets $\{d_1, d_2, \ldots, d_n\} \subseteq D$, one has

$$\mathrm{cl}(\mathrm{conv}\{\mathrm{d}_1, \mathrm{d}_2, \ldots, \mathrm{d}_\mathrm{n}\} \cap \mathrm{D}) = \mathrm{conv}\{\mathrm{d}_1, \mathrm{d}_2, \ldots, \mathrm{d}_\mathrm{n}\}.$$

To prove the existence results in what follows, the well-known intersection theorem by Ky Fan [11] is used.

Theorem 2.10. *Let K be a nonempty subset of a Hausdorff topological vector space X and $\Gamma : K \rightrightarrows X$ be a set-valued mapping with closed values such that*

(i) *Γ is a KKM mapping, that is, for any $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in K$*

$$\mathrm{conv}\{x_1, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} \Gamma(x_i);$$

(ii) *there exists a nonempty compact convex subset B of K such that $\cap_{x \in B} \Gamma(x)$ is compact.*

Then $\cap_{x \in K} \Gamma(x) \neq \emptyset$.

2.2. **Q-selected preserving $\mathbb{R}^*_-$-intersection ($\mathbb{R}^*_-$-inclusion) set-valued mappings.** In this subsection, a notion of q-s-s-d solution for set-valued equilibrium problems, which have an important role in establishing existence results for nonconvex set-valued equilibrium problems, is presented.

Definition 2.11. Let K be a nonempty subset of X, let Q be a q-s-s-d subset of K, and let $F : K \times K \rightrightarrows \mathbb{R}$ be a set-valued mapping. An element $\bar{x} \in K$ is a q-s-s-d solution of SEP (WSEP), if

$$F(\bar{x}, y) \subseteq \mathbb{R}_+, \quad \forall y \in Q,$$
$$F(\bar{x}, y) \cap \mathbb{R}_+ \neq \emptyset, \quad \forall y \in Q,$$

respectively. The set of all q-s-s-d solutions of SEP (WSEP) is denoted by $S^Q(F, K)$ (respectively, $S^Q_W(F, K)$).

An important question arises about sufficient conditions satisfying the relationships $S^Q(F, K) \subseteq S(F, K)$ and $S^Q_W(F, K) \subseteq S_W(f, K)$. To explore this way, the following notions are introduced.

Definition 2.12. Let K be a nonempty subset of X, let $Q \subseteq K$, and let $G : K \rightrightarrows \mathbb{R}$ be a set-valued mapping. We say that G is Q-selected preserving $\mathbb{R}^*_-$-intersection on K, if the following implication holds:

$$\text{if } G(x) \cap \mathbb{R}^*_- \neq \emptyset \text{ for some } x \in K \text{ then there exists } q \in Q \text{ such that } G(q) \cap \mathbb{R}^*_- \neq \emptyset.$$

The set-valued mapping G is called Q-selected preserving $\mathbb{R}^*_-$-inclusion on K, if the following implication holds:

$$\text{if } G(x) \subseteq \mathbb{R}^*_- \text{ for some } x \in K \text{ then there exists } q \in Q \text{ such that } G(q) \subseteq \mathbb{R}^*_-.$$

Example 2.13. Let $X := \mathbb{R}$ and $K := \mathbb{Q}$. Consider the mapping $G : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by $G(x) = [\sin x, |x| + 1]$. Then G is Q-selected preserving $\mathbb{R}^*_-$-intersection on K, where $Q = [\frac{-\pi}{2}, \frac{\pi}{2}] \cap \mathbb{Q}$. In fact if $G(x) = [\sin x, |x| + 1] \cap \mathbb{R}^*_- \neq \emptyset$, then for $q = \frac{-\pi}{2}$, we have $G(\frac{-\pi}{2}) = [-1, \frac{\pi}{2} + 1] \cap \mathbb{R}^*_- \neq \emptyset$.

Now, consider also the mapping $H : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by $H(x) = [-|x| - 1, \sin x]$. It is easy to see that H is Q-selected preserving $\mathbb{R}^*_-$-inclusion on K, where $Q = [\frac{-\pi}{2}, \frac{\pi}{2}] \cap \mathbb{Q}$. According to the definitions of G and H, we conclude that Q-selected preserving $\mathbb{R}^*_-$-intersection mappings and Q-selected preserving $\mathbb{R}^*_-$-inclusion mappings do not imply each other.

In the sequel, a result that highlights a large class of set-valued mappings is presented, which are Q-selected preserving $\mathbb{R}^*_-$-intersection on K, when Q is dense in the convex set K.

Recall that hemicontinuity of a set-valued mapping $G : K \rightrightarrows \mathbb{R}$ is continuity along straight lines:

Let K be a nonempty convex subset of X. We say that G is upper hemicontinuous at x_0 if the map $t \mapsto G((1 - t)x_0 + tx)$ is upper semicontinuous at 0 for every $x \in K$. Similarly, g is lower hemicontinuous at $x_0 \in K$ if the map $t \mapsto g((1 - t)x_0 + tx)$ is lower semicontinuous at 0 for every $x \in K$.

Notice that G is lower (resp. upper) hemicontinuous on $K \setminus Q$ if G is lower (resp. upper) hemicontinuous at every $x_0 \in K \setminus Q$.

Proposition 2.14. *Let K be a nonempty convex subset of X, and let Q be dense in K. Assume that the set-valued mapping $G : K \rightrightarrows \mathbb{R}$ is lower hemicontinuous on $K \setminus Q$. Then G is Q-selected preserving $\mathbb{R}^*_-$-intersection on K.*

Proof. Assume that there exists $x_0 \in K \backslash Q$ such that $G(x_0) \cap \mathbb{R}^*_- \neq \emptyset$. It follows from lower hemicontinuity of G at x_0, for every $x \in K$ there exists $\delta_x > 0$ such that $G((1-t)x_0 + tx) \cap \mathbb{R}^*_- \neq \emptyset$, where $t \in [0, \delta_x[$. Since Q is dense in K, there exists $x' \in K$ satisfying $]x_0, (1-t)x_0 + tx'[\cap Q \neq \emptyset$, where $|t| < \delta_{x'}$. Let $z = (1-t_0)x_0 + t_0 x' \in Q$ with $|t_0| < \delta_{x'}$. This implies that $G(z) \cap \mathbb{R}^*_- \neq \emptyset$ which completes the proof. □

Proposition 2.15. *Let K be a nonempty convex subset of X, and let Q be dense in K. Assume that the set-valued mapping $G : K \rightrightarrows \mathbb{R}$ is upper hemicontinuous on $K \setminus Q$. Then G is Q-selected preserving $\mathbb{R}^*_-$-inclusion on K.*

Proof. The proof is similar to the one in Proposition 2.14. □

Remark 2.16. When Q is a self-segment-dense subset of K, Proposition 2.14 is valid (since self-segment-dense sets are dense in K and K is a convex set). Since lower (resp. upper) hemicontinuity of G is weaker than lower (resp. upper) semicontinuity of a set-valued mapping G, being Q-selected preserving $\mathbb{R}^*_-$- intersection on K (resp. Q-selected preserving $\mathbb{R}^*_-$-inclusion on K) is weaker than lower (resp. upper) semicontinuity on $K \setminus Q$, when Q is dense and convex.

The following lemma makes a relationship between q-s-s-d solutions and solutions of a SEP as well as WSEP.

Lemma 2.17. *Let Q be a q-s-s-d subset of K, and let $F : K \times K \rightrightarrows \mathbb{R}$ be a set-valued bifunction. If for every $x \in K$,*

(i) *$F(x, \cdot)$ is Q-selected preserving $\mathbb{R}^*_-$-intersection on K, then $S^Q(F, K) \subseteq S(F, K)$;*
(ii) *$F(x, \cdot)$ is Q-selected preserving $\mathbb{R}^*_-$-inclusion on K, then $S^Q_W(F, K) \subseteq S_W(F, K)$.*

Proof. The proof of (i) (resp. (ii)) follows directly from the definition of Q-selected preserving $\mathbb{R}^*_-$-intersection (respectively, Q-selected preserving $\mathbb{R}^*_-$-inclusion) set-valued mappings. □

In Lemma 2.22 in [10], a condition which guarantees the inclusion $S^D(F, K) \subseteq S(F, K)$, where D is locally segment-dense in K, has been presented.

Lemma 2.18 ([10])**.** *Let K be a convex subset of X, let D be a locally segment-dense set in K and let $F : K \times K \rightrightarrows \mathbb{R}$ be a set-valued mapping. If Condition (2) presented in the following is satisfied, then $S^D(F, K) \subseteq S(F, K)$.*

Condition (1): if for every $x, y \in K$ with $F(x, y) \cap \mathbb{R}^*_- \neq \emptyset$, there exists $z \in D \cap]x, y[$ such that $F(x, z) \cap \mathbb{R}^*_- \neq \emptyset$.

Remark 2.19. One can easily verify that being Q-selected preserving $\mathbb{R}^*_-$-intersection on K with respect to the second variable is less restrictive than Condition (1).

3. Existence of solutions

In this section, existence results for set-valued equilibrium problems in real Hausdorff topological vector spaces without convexity assumption on the whole domain are established. It is worth mentioning Lemma 2.9 is a useful tool to prove the results. Throughout this section, we assume that K is a nonempty subset of X.

In the next theorem, assumptions which guarantee the existence of solutions of set-valued equilibrium problems, are presented.

Theorem 3.1. *Let K be a nonempty subset of X, and let Q be a q-s-s-d subset of K. Let $F : K \times K \rightrightarrows \mathbb{R}$ be a set-valued mapping satisfying the following conditions:*

(i) *for every* $x \in Q$, $F(x,x) \subseteq \mathbb{R}_+$;
(ii) *for every* $x \in Q$, $F(x,\cdot)$ *is lower quasiconvex on* Q;
(iii) *there exist a nonempty compact set* $K_0 \subseteq K$ *and* $y_0 \in D$ *such that*

$$F(x,y_0) \cap \mathbb{R}^*_- \neq \emptyset, \quad \forall x \in K \setminus K_0;$$

(iv) F *is* 0*-transfer lower semicontinuous on* $K_0 \times Q$;
(v) *for every* $x \in K$, $F(x,\cdot)$ *is* Q*-selected preserving* $\mathbb{R}^*_-$*-intersection on* K.

Then $S(F,K) \neq \emptyset$.

Proof. Consider the set-valued mapping $G : Q \rightrightarrows K$ by

$$G(y) = \{x \in K : F(x,y) \subseteq \mathbb{R}_+\}.$$

To prove $S^D(F,K) \neq \emptyset$, we show that $\underset{y\in Q}{\cap} G(y) \neq \emptyset$.

First, we justify that $\underset{y\in Q}{\cap} \mathrm{cl(G)(y)} \neq \emptyset$, where the set-valued mapping $\mathrm{cl(G) : Q \rightrightarrows clK}$ is defined by $\mathrm{cl(G)(y) = cl(G(y))}$. Clearly, $\mathrm{cl(G)(y)}$ is closed for every $y \in Q$. Furthermore, the coercivity condition (iii) implies that $\mathrm{cl(G)(y_0)}$ is compact. To show that $\mathrm{cl(G)}$ is a KKM mapping, let $y_1, \ldots, y_n$ be finite elements in Q and $t_1, \ldots, t_n \in \mathbb{R}_+$ be such that $\sum_{i=1}^n t_i = 1$ and $\sum_{i=1}^n t_i y_i \in Q$. Because of condition (i) and the lower quasiconvexity of $F(\sum_{i=1}^n t_i y_i, \cdot)$ on Q

$$\bigcap_{i=1}^n F(\sum_{i=1}^n t_i y_i, y_i) \subseteq F(\sum_{i=1}^n t_i y_i, \sum_{i=1}^n t_i y_i) + \mathbb{R}_+ \subseteq \mathbb{R}_+.$$

The latter implies that there exists $j \in \{1,2,\ldots,n\}$ such that $F(\sum_{i=1}^n t_i y_i, y_j) \subseteq \mathbb{R}_+$. Thus,

$$\mathrm{conv\{y_1, \ldots, y_n\} \cap Q \subseteq \cup_{i=1}^n G(y_i)},$$

and then,

$$\mathrm{cl(conv\{y_1, \ldots, y_n\} \cap Q) \subseteq cl(\cup_{i=1}^n G(y_i)) = \cup_{i=1}^n cl(G)(y_i)}.$$

Applying Lemma 2.9, we have

$$\mathrm{conv\{y_1, \ldots, y_n\} \subseteq \cup_{i=1}^n cl(G)(y_i)}.$$

The latter yields $\mathrm{cl(G)}$ is a KKM mapping. Now, it follows from Theorem 2.10 that $\underset{y\in Q}{\cap} \mathrm{cl(G(y))} \neq \emptyset$. Using assumption (iii), $G(y_0) \subseteq K_0$ and therefore

$$\underset{y\in Q}{\cap} \mathrm{cl(G)(y)} = (\underset{\mathrm{y\in Q}}{\cap} \mathrm{cl(G)(y)}) \cap \mathrm{K_0} = \underset{\mathrm{y\in Q}}{\cap} (\mathrm{cl(G)(y) \cap K_0}).$$

According to 0-transfer lower semicontinuity of F on $K_0 \times Q$, we deduce that

$$\underset{y\in Q}{\cap} (\mathrm{cl(G)(y) \cap K_0}) = \underset{\mathrm{y\in Q}}{\cap} (\mathrm{G(y) \cap K_0}).$$

We justify the latter claim: Let $x \notin \underset{y\in D}{\cap} (G(y) \cap K_0)$ and $x \in K_0$. There is $y_0 \in Q$ such that $x \notin G(y_0)$ which implies that $F(x,y_0) \cap]-\infty, 0[\neq \emptyset$. Since F is 0-transfer lower semicontinuous on $K_0 \times Q$, there exist an element $y' \in Q$ and a neighborhood U of x such that $F(z,y') \cap]-\infty, 0[\neq \emptyset$ for all $z \in U \cap K_0$. This yields $x \notin \mathrm{cl(G)(y')}$ and thus, $x \notin \underset{y\in Q}{\cap} (\mathrm{cl(G)(y) \cap K_0})$.

Hence $\underset{y\in Q}{\cap} G(y) = \underset{y\in Q}{\cap} (G(y) \cap K_0) \neq \emptyset$ which shows that $S^Q(F,K) \neq \emptyset$. Finally, It follows from assumption (v) and Lemma 2.17 that $S(F,K) \neq \emptyset$. ∎

The following corollary is derived from Theorem 3.1.

Corollary 3.2. *Let K be a convex subset of X, let Q be self-segment-dense in K, and let $F : K \times K \rightrightarrows \mathbb{R}$ be a set-valued mapping satisfying the following conditions:*

(i) *for every $x \in K$, $F(x, x) = \{0\}$;*
(ii) *for every $x \in D$, $F(x, \cdot)$ is lower quasiconvex on D;*
(iii) *there exist a nonempty compact set $K_0 \subseteq K$ and $y_0 \in D$ such that*

$$F(x, y_0) \cap \mathbb{R}^*_- \neq \emptyset, \quad \forall x \in K \setminus K_0;$$

(iv) *F is 0-transfer lower semicontinuous on $K_0 \times D$;*
(v) *for every $x \in K$, $F(x, \cdot)$ is lower hemicontinuous on $K \setminus Q$.*

Then $S(F, K) \neq \emptyset$.

Proof. From the assumptions (v) and Proposition 2.14, it follows that for every $x \in K$, $F(x, \cdot)$ is Q-selected preserving $\mathbb{R}^*_-$-intersection on K. Now, by Theorem 3.1 the desired result is derived. □

Remark 3.3. Theorem 3.1 extends Theorem 3.5 of [10] in the following aspects:

(a) The assumptions of Theorem 3.1 are imposed on a q-s-s-d subset Q of K, while the assumptions of Theorem 3.5 of [10] are imposed on a locally segment-dense subset of the convex set K. Notice that the class of the q-s-s-d sets is essentially larger than that locally segment-dense sets.
(b) It is not required for the set K to be convex;
(c) According to Remark 2.19, Condition (v) of Theorem 3.1 is weaker than Condition (v) used in Theorem 3.5 in [10].
(d) It is worth mentioning that Theorem 4.1 in [6] can be derived from Corollary 3.2.

The following result for weak set-valued equilibrium problems generalizes both Theorem 4.1 in [6] and Theorem 3.7 of [10] similar to the aspects mentioned in Remark 3.3.

Theorem 3.4. *Let K be a nonempty subset of X, and let Q be a q-s-s-d subset of K. Let $F : K \times K \rightrightarrows \mathbb{R}$ be a set-valued mapping satisfying the following conditions:*

(i) *for every $x \in Q$, $F(x, x) \cap \mathbb{R}_+ \neq \emptyset$;*
(ii) *for every $x \in Q$, $F(x, \cdot)$ is upper quasiconvex on Q;*
(iii) *there exist a nonempty compact set $K_0 \subseteq K$ and $y_0 \in Q$ such that*

$$F(x, y_0) \subseteq \mathbb{R}^*_-, \quad \forall x \in K \setminus K_0;$$

(iv) *F is 0-transfer upper semicontinuous on $K_0 \times Q$;*
(vi) *for every $x \in K$, $F(x, \cdot)$ is Q-selected preserving $\mathbb{R}^*_-$-inclusion on K.*

Then $S_W(F, K) \neq \emptyset$.

Proof. Consider the set-valued mapping $G_W : Q \rightrightarrows K$ by

$$G_W(y) = \{x \in K : F(x, y) \cap \mathbb{R}_+ \neq \emptyset\}.$$

To prove $S^D_W(F, K) \neq \emptyset$, we show that $\bigcap_{y \in Q} G_W(y) \neq \emptyset$.

First, we justify that $\bigcap_{y \in Q} \mathrm{cl}(\mathrm{G_W})(\mathrm{y}) \neq \emptyset$, where the set-valued mapping $\mathrm{cl}(\mathrm{G_W}) : \mathrm{Q} \rightrightarrows \mathrm{clK}$ is defined by $\mathrm{cl}(\mathrm{G_W})(\mathrm{y}) = \mathrm{cl}(\mathrm{G_W}(\mathrm{y}))$. Clearly, $\mathrm{cl}(\mathrm{G_W})(\mathrm{y})$ is closed for every $y \in Q$. Furthermore, the coercivity condition (iii) implies that $\mathrm{cl}(\mathrm{G_W})(\mathrm{y_0})$ is compact. To show that $\mathrm{cl}(\mathrm{G_W})$ is a KKM mapping, let $y_1, \ldots, y_n$ be finite elements in Q and $t_1, \ldots, t_n \in \mathbb{R}_+$ be such that

$\sum_{i=1}^n t_i = 1$ and $\sum_{i=1}^n t_i y_i \in Q$. Because of condition (i) and the upper quasiconvexity of $F(\sum_{i=1}^n t_i y_i, \cdot)$ on Q

$$F(\sum_{i=1}^n t_i y_i, \sum_{i=1}^n t_i y_i) - \mathbb{R}_+ \subseteq \bigcup_{i=1}^n F(\sum_{i=1}^n t_i y_i, y_i) - \mathbb{R}_+.$$

The latter implies that there exists $j \in \{1, 2, \ldots, n\}$ such that $F(\sum_{i=1}^n t_i y_i, y_j)) \cap \mathbb{R}_+ \neq \emptyset$. Thus,

$$\mathrm{conv}\{y_1, \ldots, y_n\} \cap Q \subseteq \cup_{i=1}^n G_W(y_i),$$

and then,

$$\mathrm{cl}(\mathrm{conv}\{y_1, \ldots, y_n\} \cap Q) \subseteq \mathrm{cl}(\cup_{i=1}^n G_W(y_i)) = \cup_{i=1}^n \mathrm{cl}(G_W)(y_i).$$

Applying Lemma 2.9, we have

$$\mathrm{conv}\{y_1, \ldots, y_n\} \subseteq \cup_{i=1}^n \mathrm{cl}(G_W)(y_i).$$

The latter yields $\mathrm{cl}(G_W)$ is a KKM mapping. Now, it follows from Theorem 2.10 that $\bigcap_{y\in Q} \mathrm{cl}(G_W(y)) \neq \emptyset$. Using assumption (iii), $G_W(y_0) \subseteq K_0$ and therefore

$$\bigcap_{y\in Q} \mathrm{cl}(G_W)(y) = (\bigcap_{y\in Q} \mathrm{cl}(G_W)(y)) \cap K_0 = \bigcap_{y\in Q} (\mathrm{cl}(G_W)(y) \cap K_0).$$

According to 0-transfer upper semicontinuity of F on $K_0 \times Q$, we deduce that

$$\bigcap_{y\in Q} (\mathrm{cl}(G_W)(y) \cap K_0) = \bigcap_{y\in Q} (G_W(y) \cap K_0).$$

We justify the latter claim: Let $x \notin \bigcap_{y\in D} (G_W(y) \cap K_0)$ and $x \in K_0$. There is $y_0 \in Q$ such that $x \notin G(y_0)$ which implies that $F(x, y_0) \subseteq]-\infty, 0[$. Since F is 0-transfer upper semicontinuous on $K_0 \times Q$, there exist an element $y' \in Q$ and a neighborhood U of x such that $F(z, y') \subseteq]-\infty, 0[$ for all $z \in U \cap K_0$. This yields $x \notin \mathrm{cl}(G_W)(y')$ and thus, $x \notin \bigcap_{y\in Q} (\mathrm{cl}(G_W)(y) \cap K_0)$.

Hence $\bigcap_{y\in Q} G_W(y) = \bigcap_{y\in Q} (G_W(y) \cap K_0) \neq \emptyset$ which shows that $S_W^Q(F, K) \neq \emptyset$. Finally, it follows from assumption (v) and part (b) of Lemma 2.17 that $S_W(F, K) \neq \emptyset$. □

Using Theorem 3.4 and Proposition 2.15, the next corollary establishes an existence result for a weak set-valued equilibrium problem.

Corollary 3.5. *Let K be a convex subset of X, and let Q be self-segment-dense in K. Let $F : K \times K \rightrightarrows \mathbb{R}$ be a set-valued mapping satisfying the following conditions:*

(i) *for every $x \in K$, $F(x, x) = \{0\}$;*
(ii) *for every $x \in Q$, $F(x, \cdot)$ is upper quasiconvex on Q;*
(iii) *there exist a nonempty compact set $K_0 \subseteq K$ and $y_0 \in Q$ such that*

$$F(x, y_0) \subseteq \mathbb{R}_-^*, \quad \forall x \in K \setminus K_0;$$

(iv) *F is 0-transfer upper semicontinuous on $K_0 \times Q$;*
(vi) *for every $x \in K$, $F(x, \cdot)$ is upper hemicontinuous on $K \setminus Q$.*

Then $S_W(F, K) \neq \emptyset$.

Proof. According to Proposition 2.15, for every $x \in K$, $F(x, \cdot)$ is Q-selected preserving $\mathbb{R}_-^*$-inclusion on K. Now using Theorem 3.4, the proof is complete. □

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