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Research Paper

COUPLED COINCIDENCE POINT RESULTS FOR MAPPINGS WITHOUT MIXED MONOTONE PROPERTY IN PARTIALLY ORDERED G-METRIC SPACES

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Abstract. In this paper, we prove some coupled fixed point theorems for nonlinear contractive mappings which doesn't have the mixed monotone property, in the context of partially ordered G-metric spaces. Hence, these results can be applied in a much wider class of problems. Our results improve the result of D. Dorić, Z. Kadelburg and S. Radenović [Appl. Math. Lett. (2012)]. We also present two examples to support these new results.

MSC(2010): 47H10; 54H25.
Keywords: G-metric space, coupled common fixed point, mixed monotone property.

1. Introduction and Preliminaries

The metric fixed point theory is very important and useful in Mathematics. It can be applied in various areas, such as: variational inequalities, optimization, approximation theory, etc. In many recent publications in the field of fixed point theory auxiliary functions are used to generalize the contractive conditions on the mappings defined on various spaces, see for example [11, 13, 15, 16, 30, 31].

G-metric spaces have been introduced by Mustafa and Sims in [20, 21]. This is a generalization of metric spaces in which every triplet of elements is assigned to a non-negative real number. Fixed point theory in this space was initiated in [17]. After that several fixed point results were proved in this spaces, see for example [3, 4, 9, 18, 19, 22, 25].

Recently, many results on fixed point problems have been considered in partially ordered probabilistic metric spaces [10] and in partially ordered G-metric spaces [8, 25]. In [14], coupled fixed point results in partially ordered metric spaces were established by Bhaskar and Lakshmikantham. In [1, 2, 6, 7, 8, 14, 23, 24, 26, 27, 28, 29] several coupled fixed point and coincidence point results are presented.

The aim of this paper is to establish coupled coincidence and coupled common fixed point results for mappings without mixed g-monotone property in partially ordered G-metric spaces. We also give some examples and applications. Before stating and proving our results, we recall some definitions and properties in G-metric spaces that are used in this paper.

Definition 1.1 ([21])**.** Let X be a non-empty set, $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:
$(G1)$ $G(x, y, z) = 0$ if $x = y = z$.

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$(G2)$ $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$.
$(G3)$ $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$.
$(G4)$ $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$ (symmetry in all three variables).
$(G5)$ $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).
Then the function G is called a generalized metric, or, more specially, a G-metric on X, and the pair (X,G) is called a G-metric space.

Definition 1.2 ([21])**.** Let (X,G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X. We say that $\{x_n\}$ is G-convergent to $x \in X$ if $\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$, that is, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < \epsilon$, for all $n,m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $lim\ x_n = x$.

Lemma 1.3 ([21])**.** *Let (X,G) be a G-metric space. The following are equivalent:*
(1) $\{x_n\}$ *is G-convergent to x.*
(2) $G(x_n,x_n,x) \to 0$ *as* $n \to \infty$.
(3) $G(x_n,x,x) \to 0$ *as* $n \to \infty$.
(4) $G(x_n,x_m,x) \to 0$ *as* $n,m \to \infty$.

Definition 1.4 ([21])**.** Let (X,G) be a G-metric space, A sequence $\{x_n\}$ is called a G-Cauchy sequence if, for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \epsilon$ for all $m,n,l \in \mathbb{N}$, that is, $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to \infty$.

Lemma 1.5 ([20])**.** *Let (X,G) be a G-metric space. Then the following are equivalent:*
(1) *the sequence $\{x_n\}$ is G-Cauchy*
(2) *for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_m) < \epsilon$ for all $m,n \geq \mathbb{N}$.*

Lemma 1.6 ([8])**.** *Let (X,G) be a G-metric space. A mapping $f : X \to X$ is G-continuous at $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G-convergent to $f(x)$.*

Definition 1.7 ([21])**.** A G-metric space (X,G) is called G-complete if every G-Cauchy sequence is G-convergent in (X,G).

Definition 1.8 ([20])**.** A G-metric space (X,G) is called a symmetric G-metric space if $G(x,y,y) = G(y,x,x)$ for all $x,y \in X$.

Definition 1.9 ([21])**.** Let (X,G) be a G-metric space. A mapping $F : X \times X \to X$ is said to be continuous if for any two G-convergent sequences $\{x_n\}$ and $\{y_n\}$ converging to x and y respectively, then $\{F(x_n,y_n)\}$ is G-convergent to $F(x,y)$.

In 2006 the concept of a mixed monotone property has been introduced by Bhaskar and Lakshmikantham in [5].

Definition 1.10 ([5])**.** Let $(X,\preceq)$ be a partially ordered set. A mapping $F : X \times X \to X$ is said to have mixed monotone property if $F(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y; that is, for any $x,y \in X$,

$$x_1, x_2 \in X, x_1 \preceq x_2 \text{ implies } F(x_1,y) \preceq F(x_2,y),$$
$$y_1, y_2 \in X, y_1 \preceq y_2 \text{ implies } F(x,y_2) \preceq F(x,y_1).$$

In 2009 Lakshmikantham and Ćirić in [14] introduced the concept of a g-mixed monotone mapping.

Definition 1.11 ([14]). Let $(X, \preceq)$ be a partially ordered set. Let us consider mappings $F : X \times X \to X$ and $g : X \to X$. The map F is said to have mixed g-monotone property if $F(x, y)$ is monotone g-non-decreasing in x and is monotone g-non-increasing in y; that is, for any $x, y \in X$,

$$x_1, x_2 \in X, gx_1 \preceq gx_2 \ \to \ F(x_1, y) \preceq F(x_2, y),$$
$$y_1, y_2 \in X, gy_1 \preceq gy_2 \ \to \ F(x, y_2) \preceq F(x, y_1).$$

An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$ [5].

Also An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$ [14].

In 2011 Choudhury and Maity established some coupled fixed point results for the mixed monotone mappings in [8].

Theorem 1.12 ([8, Theorem 2.1]). *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that (X, G) is a complete G-metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists $k \in [0, 1)$ such that*

$$G(F(x, y), F(u, v), F(z, w)) \ \leq \ \frac{k}{2}[G(x, u, z) + G(y, v, w)], \tag{1.1}$$

for all $x, y, u, v, z, w \in X$ with $x \succeq u \succeq z$ and $y \preceq v \preceq w$ where either $u \neq z$ or $v \neq w$. If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point in X.

Definition 1.13 ([14]). We say that two mappings $F : X \times X \to X$ and $g : X \to X$ are commutative if

$$g(F(x, y)) = F(gx, gy) \ \forall x, y \in X.$$

The following definition which was given by Dorić, Kadelburg and Radenović in [12], will be used in this paper.

Definition 1.14. Let (X, d) be a metric space and let $g : X \to X$, $F : X \times X \to X$. The mappings g and F are said to be compatible if

$$\lim_{n\to\infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0,$$

and

$$\lim_{n\to\infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0,$$

hold whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n$.

Definition 1.15 ([12]). If elements x, y of a partially ordered set $(X, \preceq)$ are comparable (i.e., $x \preceq y$ or $y \preceq x$ holds) we will write $x \asymp y$. Let $g : X \to X$ and $F : X \times X \to X$ be two mappings. We will consider the following condition, if $x, y, u, v \in X$ are such that

$$gx \asymp F(x, y) = gu \text{ then } F(x, y) \asymp F(u, v). \tag{1.2}$$

In particular, when $g = I_X$, it reduces to for all $x, y, v \in X$ if

$$x \asymp F(x, y) \text{ then } F(x, y) \asymp F(F(x, y), v). \tag{1.3}$$

In 2012, Dorić, Kadelburg and Radenović have been shown in a simple example that these conditions may be satisfied when F does not have the g-mixed monotone property. They also established some coupled fixed point results for mappings without mixed monotone property in [12].

Theorem 1.16 ([12, Theorem 2.3])**.** *Let $(X,d,\preceq)$ be a complete partially ordered metric space and let $g : X \to X$ and $F : X \times X \to X$. Suppose that the following hold:*

(i) g is continuous and $g(X)$ is closed;
(ii) $F(X \times X) \subseteq g(X)$ and g and F are compatible;
(iii) g and F satisfy property (1.2);
(iv) there exist $x_0, y_0 \in X$ such that $gx_0 \asymp F(x_0,y_0)$ and $gy_0 \asymp F(y_0,x_0)$;
(v) there exist $k \in [0,1)$ such that for all $x,y,u,v \in X$ satisfying $gx \asymp gu$ and $gy \asymp gv$,

$$d(F(x,y),F(u,v)) \quad \leq \quad k \ \max\{d(gx,gu),d(gy,gv)\},$$

hold true;
(vi) F is a continuous or if $x_n \to x$ in X, then $x_n \asymp x$ for n sufficiently large.

Then there exist $u,v \in X$ such that $gu = F(u,v)$ and $gv = F(v,u)$, that is, g and F have a coupled coincidence point.

The main purpose of this paper is to prove some coupled fixed point theorems for nonlinear contractive mappings which don't have the mixed monotone property in the context of partially ordered G-metric spaces. We also present two examples to support our new results.

2. Main Results

We start this section with a new definition.

Definition 2.1. Let X be a G-metric space let $g : X \to X$, $F : X \times X \to X$ be two mappings. The mappings g and F are said to be G-compatible if

$$\lim_{n\to\infty} G(gF(x_n,y_n),F(gx_n,gy_n),F(gx_n,gy_n)) = 0,$$

and

$$\lim_{n\to\infty} G(gF(y_n,x_n),F(gy_n,gx_n),F(gy_n,gx_n)) = 0,$$

hold whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n,y_n) = \lim_{n\to\infty} gx_n$ and $\lim_{n\to\infty} F(y_n,x_n) = \lim_{n\to\infty} gy_n$.

Remark 2.2. In this paper, if the elements x,y,z of a partially ordered set $(X,\preceq)$ are comparable (i.e., $x \preceq y \preceq z$ or $z \preceq y \preceq x$ holds) we will write $x \asymp y \asymp z$.

The following theorem is the first main result of this paper.

Theorem 2.3. *Let $(X,\preceq)$ be a partially ordered set and G be a G-metric on X such that (X,G) is a complete G-metric space. Suppose that for two mappings $g : X \to X$ and $F : X \times X \to X$ there exists $k \in [0,1)$ such that*

$$G(F(x,y),F(u,v),F(z,w)) \leq k\ max\{G(gx,gu,gz),G(gy,gv,gw)\}, \tag{2.1}$$

for all $x,y,u,v,z,w \in X$ with $gx \asymp gu \asymp gz$ and $gy \asymp gv \asymp gw$. Suppose also that g is continuous, $g(X)$ is closed, $F(X \times X) \subseteq g(X)$, g and F are G-compatible and F and g satisfy property (1.2). Suppose that either

(a) F *is continuous,*
or
(b) if $x_n \to x$ *in* X*, then* $x_n \asymp x$ *for* n *sufficiently large.*

If there exist $x_0, y_0 \in X$ *such that* $gx_0 \asymp F(x_0, y_0)$ *and* $gy_0 \asymp F(y_0, x_0)$*, then* g *and* F *have a coupled coincidence point, that is, there exist* $u, v \in X$ *such that* $gu = F(u, v)$ *and* $gv = F(v, u)$*. Moreover, if for every two coupled coincidence point* (x, y) *and* (u, v) *if* (x, y) *and* (u, v) *are comparable (*$x \asymp u, y \asymp v$ *), then* $gx = gu$ *and* $gy = gv$*.*

Proof. Let $x_0, y_0 \in X$ such that $gx_0 \asymp F(x_0, y_0)$ and $gy_0 \asymp F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Again since $F(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Continuing in this way we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all $n \geq 0$,

$$g(x_{n+1}) = F(x_n, y_n) \text{ and } g(y_{n+1}) = F(y_n, x_n). \tag{2.2}$$

Now we prove that for all $n \geq 0$

$$g(x_n) \asymp g(x_{n+1}) \text{ and } g(y_n) \asymp g(y_{n+1}). \tag{2.3}$$

We shall use the mathematical induction. Let $n = 0$. Since $gx_0 \asymp F(x_0, y_0)$ and $gy_0 \asymp F(y_0, x_0)$, in view of $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$, we have $g(x_0) \asymp g(x_1)$ and $g(y_0) \asymp g(y_1)$, that is, (2.3) hold for $n = 0$. We presume that (2.3) hold for some $n > 0$. As F and g have property (1.2) and $g(x_n) \asymp g(x_{n+1})$, $g(y_n) \asymp g(y_{n+1})$, from (2.2), we get

$$g(x_{n+1}) = F(x_n, y_n) \asymp F(x_{n+1}, y_{n+1}) = g(x_{n+2}), \tag{2.4}$$

and

$$g(y_{n+1}) = F(y_n, x_n) \asymp F(y_{n+1}, x_{n+1}) = g(y_{n+2}). \tag{2.5}$$

Then from (2.4) and (2.5) we obtain

$$g(x_{n+1}) \asymp g(x_{n+2}) \text{ and } g(y_{n+1}) \asymp (y_{n+2}). \tag{2.6}$$

Thus by the mathematical induction, we conclude that (2.3) holds for all $n \geq 0$.

If for some n, we have $(gx_{n+1}, gy_{n+1}) = (gx_n, gy_n)$, then $F(x_n, y_n) = gx_n$ and $F(y_n, x_n) = gy_n$, that is, F and g have a coincidence point. So from now on, we assume $(gx_{n+1}, gy_{n+1}) \neq (gx_n, gy_n)$, for all $n \in N$, that is, we assume that either $gx_{n+1} = F(x_n, y_n) \neq gx_n$ or $gy_{n+1} = F(y_n, x_n) \neq gy_n$.
Since $g(x_n) \asymp g(x_{n-1})$ and $g(y_n) \asymp g(y_{n-1})$, from contractive condition (2.1), we have

$$\begin{aligned} G(gx_{n+1}, gx_{n+1}, gx_n) &= G(F(x_n, y_n), F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\leq k \; max\{G(gx_n, gx_n, gx_{n-1}), G(gy_n, gy_n, gy_{n-1})\}, \end{aligned}$$

and

$$\begin{aligned} G(gy_{n+1}, gy_{n+1}, gy_n) &= G(F(y_n, x_n), F(y_n, x_n), F(y_{n-1}, x_{n-1})) \\ &\leq k \; max\{G(gy_n, gy_n, gy_{n-1}), G(gx_n, gx_n, gx_{n-1})\}, \end{aligned}$$

and hence

$$\begin{aligned} & max \; \{G(gx_{n+1}, gx_{n+1}, gx_n), G(gy_{n+1}, gy_{n+1}, gy_n)\} \\ \leq \; & k \; max\{G(gx_n, gx_n, gx_{n-1}), G(gy_n, gy_n, gy_{n-1})\}, \end{aligned}$$

for each $n \in \mathbb{N}$. By induction we get that

$$\begin{aligned} & max\ \{G(gx_{n+1}, gx_{n+1}, gx_n), G(gy_{n+1}, gy_{n+1}, gy_n)\} \\ \leq\ & k\ max\{G(gx_n, gx_n, gx_{n-1}), G(gy_{n-1}, gy_{n-1}, gy_{n-2})\} \\ \leq\ & k^2\ max\{G(gx_{n-1}, gx_{n-1}, gx_{n-2}), G(gy_n, gy_n, gy_{n-1})\} \\ \leq\ & ... \leq k^n\ max\{G(gx_1, gx_1, gx_0), G(gy_1, gy_1, gy_0)\}, \end{aligned}$$

Now, we shall show that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. For all positive integers $n, m \in \mathbb{N}$, $n < m$ we have by the rectangle inequity ($G5$ of Definition 1.1) that

$$\begin{aligned} G(gx_m, gx_m, gx_n) \ &\leq\ G(gx_m, gx_m, gx_{m-1}) + G(gx_{m-1}, gx_{m-1}, gx_{m-2}) \\ &\quad + ... + G(gx_{n+1}, gx_{n+1}, gx_n) \\ &\leq\ k^{m-1}\ max\{G(gx_1, gx_1, gx_0), G(gy_1, gy_1, gy_0)\} \\ &\quad + k^{m-2}\ max\{G(gx_1, gx_1, gx_0), G(gy_1, gy_1, gy_0)\} \\ &\quad + ... + k^n\ max\{G(gx_1, gx_1, gx_0), G(gy_1, gy_1, gy_0)\} \\ &=\ (k^{m-1} + k^{m-2} + ... + k^n) max\{G(gx_1, gx_1, gx_0), G(gy_1, gy_1, gy_0)\} \\ &=\ k^n(1 + k + ... + k^{m-n-1}) max\{G(gx_1, gx_1, gx_0), G(gy_1, gy_1, gy_0)\} \\ &<\ \frac{k^n}{1-k} max\{G(gx_1, gx_1, gx_0), G(gy_1, gy_1, gy_0)\}. \end{aligned}$$

Letting $m, n \to \infty$ in above inequality, we conclude that $\lim_{m,n\to\infty} G(gx_m, gx_m, gx_n) = 0$, and similarly $\lim_{m,n\to\infty} G(gy_m, gy_m, gy_n) = 0$. Therefore, $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences and, since $g(X)$ is closed in a complete metric space there exists $u, v \in g(X)$ such that

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} F(x_n, y_n) = u \ and\ \lim_{n\to\infty} gy_n = \lim_{n\to\infty} F(y_n, x_n) = v. \tag{2.7}$$

G-compatibility of g and F implies that

$$\lim_{n\to\infty} G(gF(x_n, y_n), F(gx_n, gy_n), F(gx_n, gy_n)) = 0, \tag{2.8}$$

and

$$\lim_{n\to\infty} d(gF(y_n, x_n), F(gy_n, gx_n), F(gy_n, gx_n)) = 0.$$

Now, suppose that assumption (a) holds. Using rectangle inequality ($G5$ of Definition 1.1) we have

$$\begin{aligned} & G(gu, F(gx_n, gy_n), F(gx_n, gy_n)) \leq G(gu, gF(x_n, y_n), gF(x_n, y_n)) \\ & + G(gF(x_n, y_n), F(gx_n, gy_n), F(gx_n, gy_n)). \end{aligned} \tag{2.9}$$

Passing to the limit in (2.9) when $n \to \infty$ and using (2.8) and continuity of g and F we get that $G(gu, F(u, v), F(u, v)) = 0$, that is, $gu = F(u, v)$. Similarly, we can show that $F(v, u) = gv$.

Finally, suppose that (b) holds. Since $gx_n \to u$ and $gy_n \to v$ and $u, v \in g(X)$. We get that $gx_n \asymp u = gx$ and $gy_n \asymp v = gy$ for some $x, y \in X$ and n sufficiently large. For such n, using (2.1) we have

$$\begin{aligned} G(F(x, y), gx, gx) &\leq G(F(x, y), gx_{n+1}, gx_{n+1}) + G(gx_{n+1}, gx, gx) \\ &= G(F(x, y), F(x_n, y_n), F(x_n, y_n)) + G(gx_{n+1}, gx, gx) \\ &\leq k \max\{G(gx, gx_n, gx_n), G(gy, gy_n, gy_n)\} + G(gx_{n+1}, gx, gx) \end{aligned}$$

Taking the limit when $n \to \infty$, we get that $G(F(x,y), gx, gx) = 0$. Hence $gx = F(x,y)$ and similarly $gy = F(y,x)$. This completes the proof. $\square$

Remark 2.4. Note that in the case (b), continuity of g and G-compatibility of (g,F) assumptions were not needed in the proof.

Remark 2.5. Theorem 2.3 remains valid if the right-hand side of condition (2.1) is replaced by $kG(gx,gu,gz) + lG(gy,gv,gw)$, for some $k,l \geq 0$ with $k+l<1$, because

$$kG(gx,gu,gz) + lG(gy,gv,gw) \leq (k+l)\max\{G(gx,gu,gz), G(gy,gv,gw)\},$$

which reduces to condition (1.1) of Theorem 1.12 if $g = I_X$ and $k = l$.

The following theorem is a direct result of Theorem 2.3.

Theorem 2.6. *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that (X,G) is a complete G-metric space. Suppose that there exist $k \in [0,1)$ and $F : X \times X \to X$ such that*

$$G(F(x,y), F(u,v), F(z,w)) \leq k\ max\{G(x,u,z), G(y,v,w)\}, \tag{2.10}$$

for all $x,y,u,v,z,w \in X$ with $x \asymp u \asymp z$ and $y \asymp v \asymp w$. Suppose also F satisfy property (1.3). Suppose that either

(a) F is continuous,
or
(b) if $x_n \to x$ in X, then $x_n \asymp x$ for n sufficiently large.

If there exist $x_0, y_0 \in X$ such that $x_0 \asymp F(x_0,y_0)$ and $y_0 \asymp F(y_0,x_0)$, then F has a coupled fixed point,that is, there exist $u,v \in X$ such that $u = F(u,v)$ and $v = F(v,u)$. Moreover, if for every two coupled fixed point (x,y) and (u,v) if (x,y) and (u,v) are comparable $(x \asymp u, y \asymp v)$, then $x = u$ and $y = v$.

Proof. Let $g = I_X$ and apply Theorem 2.3. $\square$

The following theorem is the second main result of this paper that extends Theorem 1.16.

Theorem 2.7. *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that (X,G) is a complete G-metric space. Suppose that there exist $k \in [0,1)$, $g : X \to X$ and $F : X \times X \to X$ such that*

$$G(F(x,y), F(x,y), F(u,v)) \leq k\ max\{G(gx,gx,gu), G(gy,gy,gv)\}, \tag{2.11}$$

for all $x,y,u,v \in X$ with $gx \asymp gu$ and $gy \asymp gv$. Suppose also that g is continuous, $g(X)$ is closed, $F(X \times X) \subseteq g(X)$, g and F are G-compatible and F and g satisfy property (1.2). Suppose that either

(a) F is continuous,
or
(b) if $x_n \to x$ in X, then $x_n \asymp x$ for n sufficiently large.

If there exist $x_0, y_0 \in X$ such that $gx_0 \asymp F(x_0,y_0)$ and $gy_0 \asymp F(y_0,x_0)$, then g and F have a coupled coincidence point,that is, there exist $u,v \in X$ such that $gu = F(u,v)$ and $gv = F(v,u)$. Moreover, if for every two coupled coincidence point (x,y) and (u,v) if (x,y) and (u,v) are comparable $(x \asymp u, y \asymp v)$, then $gx = gu$ and $gy = gv$.

Proof. The proof is similar to the proof of Theorem 2.3. $\square$

Remark 2.8. By defining $G(x,y,z) = \frac{d(x,y)+d(y,z)+d(z,x)}{2}$, in Theorem 2.7, we conclude Theorem 1.16.

Now we shall prove the existence and uniqueness theorem of a coupled common fixed point. If $(X, \preceq)$ is a partially ordered set, we endow the product set $X \times X$ with the partial order $\triangleright$ defined by

$$(x,y) \triangleright (u,v) \Leftrightarrow x \preceq u \text{ and } v \preceq y. \tag{2.12}$$

Theorem 2.9. *In addition to the hypotheses of Theorem 2.3, suppose that,*
(c) for every two elements $(x,y),(u,v) \in X \times X$, *there exists* $(w,z) \in X \times X$ *such that* $(F(w,z),F(z,w))$ *is comparable to both* $(F(x,y),F(y,x))$ *and* $(F(u,v),F(v,u))$.
Then F *and* g *have a unique common coincidence fixed point, that is, there exists a unique* $(p,q) \in X \times X$ *such that* $p = gp = F(p,q)$ *and* $q = gq = F(q,p)$.

Proof. From Theorem 2.3, the set of coupled coincidence is non-empty. We shall show that if (x,y) and (u,v) are coupled coincidence points, that is, if $gx = F(x,y)$, $gy = F(y,x)$, $gu = F(u,v)$ and $gv = F(v,u)$, then

$$gx = gu \text{ and } gy = gv. \tag{2.13}$$

By assumption, there exists $(w,z) \in X \times X$ such that $(F(w,z),F(z,w))$ is comparable to both $(F(x,y),F(y,x))$ and $(F(u,v),F(v,u))$. Without restriction to the generality, we can assume that

$$\begin{aligned}(F(x,y),F(y,x)) &\triangleright (F(w,z),F(z,w))\\ (F(u,v),F(v,u)) &\triangleright (F(w,z),F(z,w)).\end{aligned}$$

Put $w_0 = w$, $z_0 = z$ and choose $w_1, z_1 \in X$ such that $gw_1 = F(w_0,z_0)$ and $gz_1 = F(z_0,w_0)$. Then, similarly as in the proof of Theorem 2.3, we can inductively define sequences $\{gw_n\}$ and $\{gz_n\}$ in X by

$$gw_{n+1} = F(w_n,z_n) \text{ and } gz_{n+1} = F(z_n,w_n) \tag{2.14}$$

for $n \in \mathbb{N}$. By taking $x_0 = x_1 = x_2 = ... = x_n = x$, $y_0 = y_1 = y_2 = ... = y_n = y$, $u_0 = u_1 = u_2 = ... = u_n = u$ and $v_0 = v_1 = v_2 = ... = v_n = v$, for all $n \in \mathbb{N}$, we have:

$$gx_n = F(x,y),\ gy_n = F(y,x) \text{ and } gu_n = F(u,v),\ gv_n = F(v,u). \tag{2.15}$$

Since

$$(F(x,y),F(y,x)) = (gx_1,gy_1) = (gx,gy) \triangleright (F(w,z),F(z,w)) = (gw_1,gz_1),$$

then $gx \asymp gw_1$ and $gy \asymp gz_1$. Using that F satisfy property (1.3), one can show easily that $gx \asymp gw_n$ and $gy \asymp gz_n$ for all $n \geq 1$. Thus, from (2.1), we get

$$\begin{aligned} G(gw_{n+1},gx,gx) &= G(F(w_n,z_n),F(x,y),F(x,y))\\ &\leq k\ max\{G(gw_n,gx,gx),G(gz_n,gy,gy)\}, \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} G(gz_{n+1},gy,gy) &= G(F(z_n,w_n),F(y,x),F(y,x))\\ &\leq k\ max\{G(gz_n,gy,gy),G(gw_n,gx,gx)\}. \end{aligned} \tag{2.17}$$

From (2.16) and (2.17), we have

$$\begin{aligned} &\max\{G(gw_{n+1},gx,gx),G(gz_{n+1},gy,gy)\}\\ \leq\ &k\ max\{G(gw_n,gx,gx),G(gz_n,gy,gy)\}. \end{aligned}$$

Using induction we get

$$\begin{aligned} & \max\{G(gw_{n+1}, gx, gx), G(gz_{n+1}, gy, gy)\} \\ \leq\ & k\ max\{G(gw_n, gx, gx), G(gz_n, gy, gy)\}, \\ \leq\ & k^2\ max\{G(gw_{n-1}, gx, gx), G(gz_{n-1}, gy, gy)\}, \\ \leq\ & ... \leq k^n\ max\{G(gw_1, gx, gx), G(gz_1, gy, gy)\}. \end{aligned} \tag{2.18}$$

Letting $n \to \infty$, in (2.18) we obtain

$$\lim_{n\to\infty} \max\{G(gw_{n+1}, gx, gx), G(gz_{n+1}, gy, gy)\} = 0.$$

Consequently,

$$\lim_{n\to\infty} G(gw_{n+1}, gx, gx) = 0\ and\ \lim_{n\to\infty} G(gz_{n+1}, gy, gy) = 0. \tag{2.19}$$

Similarly, one can show that

$$\lim_{n\to\infty} G(gw_{n+1}, gu, gu) = 0\ and\ \lim_{n\to\infty} G(gz_{n+1}, gv, gv) = 0. \tag{2.20}$$

Therefore, from (2.19), (2.20) and the uniqueness of the limit, we get $gx = gu$ and $gy = gv$. So (2.13) holds.

Denote now $gx = p$ and $gy = q$, so we have that

$$gp = g(gx) = gF(x, y)\ and\ gq = g(gy) = gF(y, x). \tag{2.21}$$

By definition of the sequences $\{x_n\}$ and $\{y_n\}$ we have

$$gx_n = F(x, y) = F(x_{n-1}, y_{n-1})\ and\ gy_n = F(y, x) = F(y_{n-1}, x_{n-1}),$$

so continuity F implies

$$F(x_{n-1}, y_{n-1}) \to F(x, y)\ and\ gx_n \to F(x, y),$$

also

$$F(y_{n-1}, x_{n-1}) \to F(y, x)\ and\ gy_n \to F(y, x).$$

G-compatibility of g and F implies that

$$G(gF(x_n, y_n), F(gx_n, gy_n), F(gx_n, gy_n)) \to 0,\ n \to \infty,$$

so $gF(x, y) = F(gx, gy)$. From (2.21) we get that

$$gp = g(gx) = gF(x, y) = F(gx, gy) = F(p, q), \tag{2.22}$$

in a similar way,

$$gq = g(gy) = gF(y, x) = F(gy, gx) = F(q, p), \tag{2.23}$$

so $gp = F(p, q)$ and $gq = F(q, p)$. Thus, (p, q) is a coincidence point. Then, from (2.13) with $u = p$ and $v = q$, we have $gx = gp = p$ and $gy = gq = q$, that is,

$$gp = p\ and\ gq = q. \tag{2.24}$$

From (2.22), (2.23) and (2.24), we get

$$p = gp = F(p, q)\ and\ q = gq = F(q, p).$$

Therefore, (p, q) is a common coupled fixed point of g and F. To prove the uniqueness, assume that (x_1, x_2) is another coupled common fixed point. Then by (2.13), we have $x_1 = gx_1 = gp = p$ and $x_2 = gx_2 = gq = q$. This completes the proof. □

The following theorem extends Theorem 2.7 in [12].

Theorem 2.10. *In addition to the hypotheses of Theorem 2.7, suppose that,*
(c) for every two elements $(x,y),(u,v) \in X \times X$*, there exists* $(w,z) \in X \times X$ *such that* $(F(w,z),F(z,w))$ *is comparable to both* $(F(x,y),F(y,x))$ *and* $(F(u,v),F(v,u))$*.*
Then F *and* g *have a unique common coincidence fixed point, that is, there exists a unique* $(p,q) \in X \times X$ *such that* $p = gp = F(p,q)$ *and* $q = gq = F(q,p)$*.*

Proof. The proof is similar to the proof of Theorem 2.9. □

By definition $G(x,y,z) = \frac{d(x,y)+d(y,z)+d(z,x)}{2}$, in Theorem 2.10, we have the following result.

Corollary 2.11 (see [12, Theorem 2.7])**.** *In addition to the hypotheses of Theorem 1.16, suppose that, for every* $(x,y),(u,v) \in X \times X$*, there exists* $(w,z) \in X \times X$ *such that* $(F(w,z),F(z,w))$ *is comparable to both* $(F(x,y),F(y,x))$ *and* $(F(u,v),F(v,u))$*.*
Then F *has a unique common couple coincidence point, that is, there exists a unique* $(p,q) \in X \times X$ *such that* $p = gp = F(p,q)$ *and* $q = gq = F(q,p)$*.*

Theorem 2.12. *In addition to the hypotheses of Theorem 2.6, suppose that,*
(c) for every two elements $(x,y),(u,v) \in X \times X$*, there exists* $(w,z) \in X \times X$ *such that* $(F(w,z),F(z,w))$ *is comparable to both* $(F(x,y),F(y,x))$ *and* $(F(u,v),F(v,u))$*.*
Then F *and* g *have a unique common couple fixed point, that is, there exists a unique* $(p,q) \in X \times X$ *such that* $p = F(p,q)$ *and* $q = F(q,p)$*.*

Proof. Let $g = I_X$ and apply Theorem 2.9. □

Theorem 2.13. *In addition to the hypotheses of Theorem 2.3, if* $gx_0 \asymp gy_0$*, then* F *and* g *have a unique common coupled coincidence point* $(x,y) \in X \times X$ *such that* $gx = F(x,y) = F(y,x) = gy$*.*

Proof. Following the proof of Theorem 2.3, F and g have a coupled coincidence point. We only have to show that $gx = gy$. By Theorem 2.3, $gx_n \to u$ and $gy_n \to v$ and $u, v \in g(X)$, we get that $gx_n \asymp u = gx$ and $gy_n \asymp v = gy$ for some $x, y \in X$ and n sufficiently large. Since $gx_0 \asymp gy_0$, By using mathematical induction and mixed monotone property of F, one can show that

$$gx_n \asymp gy_n \ forall \ n \geq 0.$$

where $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$, $n = 0, 1, ...$
Thus, by (2.1) we have

$$\begin{aligned} G(gx_{n+1}, gy_{n+1}, gy_{n+1}) &= G(F(x_n, y_n), F(y_n, x_n), F(y_n, x_n)) \\ &\leq \ k \ \max\{G(gx_n, gy_n, gy_n), G(gy_n, gx_n, gx_n)\}. \end{aligned}$$

Passing to the limit when $n \to \infty$, since $gx_n \to gx$ and $gy_n \to gy$, we get $(1-k)G(gx, gy, gy) \leq 0$. So $G(gx, gy, gy) = 0$. Hence $gx = F(x,y) = F(y,x) = gy$. □

Theorem 2.14. *In addition to the hypotheses of Theorem 2.6, if* $x_0 \asymp y_0$*, then* F *has a fixed point,that is, there exist a* $x \in X$ *such that* $x = F(x,x)$*.*

Proof. Let $g = I_X$ and apply Theorem 2.13. □

3. Examples

In the section, some examples are presented.

Example 3.1. Let $X = [0, 1]$, Then $(X, \preceq)$ is a partially ordered set with the natural ordering of real numbers. let G be the G- metric on $X \times X \times X$ defined as follows:

$$G(x, y, z) = max\{|x - y|, |y - z|, |z - x|\} \ \forall \ x, y, z \in X.$$

Define $g : X \to X$ by $g(x) = x^2$ and $F : X \times X :\to X$ by

$$F(x, y) = \frac{x^2 + 3y^2}{12} \ \forall \ x, y \in X.$$

Obviously, (X, G) is a complete G-metric space, F and g satisfy property (1.2), g is continuous, $g(X)$ is closed and also $F(X \times X) \subseteq g(X)$

We will check that g and F are G-compatible. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} F(x_n, y_n) = a \ and \ \lim_{n\to\infty} gy_n = \lim_{n\to\infty} F(y_n, x_n) = b.$$

Then $\frac{a+3b}{12} = a$ and $\frac{b+3a}{12} = b$, wherefore it follows that $a = b = 0$. Now, for all $n \geq 0$, $g(x_n) = x_n^2$, $g(y_n) = y_n^2$, we have

$$\begin{aligned} & G(gF(x_n, y_n), F(gx_n, gy_n), F(gx_n, gy_n)) \\ = \ & G\Big(\big(\frac{{x_n}^2 + 3{y_n}^2}{12}\big)^2, \frac{{x_n}^4 + 3{y_n}^4}{12}, \frac{{x_n}^4 + 3{y_n}^4}{12}\Big) \\ = \ & max\Big\{\Big|\big(\frac{{x_n}^2 + 3{y_n}^2}{12}\big)^2 - \frac{{x_n}^4 + 3{y_n}^4}{12}\Big|, \Big|\frac{{x_n}^4 + 3{y_n}^4}{12} - \frac{{x_n}^4 + 3{y_n}^4}{12}\Big|, \\ & \Big|\frac{{x_n}^4 + 3{y_n}^4}{12} - \big(\frac{{x_n}^2 + 3{y_n}^2}{12}\big)^2\Big|\Big\} \to 0 \ (n \to \infty), \end{aligned}$$

and similarly, $G(gF(y_n, x_n), F(gy_n, gx_n), F(gy_n, gx_n)) \to 0$. Hence g and F are G-compatible.

Let $x_0 = 0$ and $y_0 = c > 0$ be two points in X. Then

$$\begin{aligned} g(x_0) = g(0) = 0 \leq \frac{3c^2}{12} = F(0, c) = F(x_0, y_0) \ and \\ g(y_0) = g(c) = c^2 \geq \frac{c^2}{12} = F(c, 0) = F(y_0, x_0). \end{aligned}$$

Consequently, $g(x_0) \asymp F(x_0, y_0)$ and $g(y_0) \asymp F(y_0, x_0)$.

We next verify inequality (2.1) of Theorem 2.2. We take $x, y, u, v, z, w \in X$, such that $gx \asymp gu \asymp gz$ and $gy \asymp gv \asymp gw$, that is, $x^2 \asymp u^2 \asymp z^2$ and $y^2 \asymp v^2 \asymp w^2$. Then

$$\begin{aligned}
& G(F(x,y), F(u,v), F(z,w)) \\
= \ & max\Big\{|\frac{x^2+3y^2}{12} - \frac{u^2+3v^2}{12}|, |\frac{u^2+3v^2}{12} - \frac{z^2+3w^2}{12}|, \\
& |\frac{z^2+3w^2}{12} - \frac{x^2+3y^2}{12}|\Big\} \\
\leq \ & max\Big\{\frac{1}{12}|x^2-u^2| + \frac{3}{12}|y^2-v^2|, \frac{1}{12}|u^2-z^2| + \frac{3}{12}|v^2-w^2|, \\
& \frac{1}{12}|z^2-x^2| + \frac{3}{12}|w^2-y^2|\Big\} \\
\leq \ & max\Big\{\frac{4}{12}max\{|x^2-u^2|, |y^2-v^2|\}, \frac{4}{12}max\{|u^2-z^2|, v^2-w^2|\}, \\
& \frac{4}{12}max\{|z^2-x^2|, |w^2-y^2|\}\Big\} \\
= \ & \frac{4}{12}max\Big\{max\{|x^2-u^2|, |u^2-z^2|, |z^2-x^2|\}, \\
& max\{|y^2-v^2|, |v^2-w^2|, |w^2-x^2|\}\Big\} \\
= \ & \frac{4}{12}max\{G(gx, gu, gz), G(gy, gv, gw)\}.
\end{aligned}$$

Hence the required condition of Theorem 2.3 and 2.9 are satisfied and there exists unique common coupled fixed point $(0,0)$ of the mappings g and F. Note that F does not satisfy the g-mixed monotone property. Also, g and F do not commute.

Example 3.2. Let $X = \mathbb{R}$, Then $(X, \preceq)$ is a partially ordered set with the natural ordering of real numbers. let G be the G- metric on $X \times X \times X$ defined as follows:

$$G(x,y,z) = max\{|x-y|, |y-z|, |z-x|\} \ \forall \ x,y,z \in X.$$

Define $F : X \times X :\to X$ by

$$F(x,y) = \frac{1}{6}x + \frac{1}{4}y \ \forall \ x,y \in X.$$

Obviously, (X, G) is a complete G-metric space, F satisfys property (1.3) and F is continuous.

Also Let $x_0 = 0$ and $y_0 = c > 0$ be two points in X. Then

$$\begin{aligned}
x_0 = 0 \leq \frac{c}{5} = F(0,c) = F(x_0, y_0) \ and \\
y_0 = c \geq \frac{c}{3} = F(c,0) = F(y_0, x_0).
\end{aligned}$$

Consequently, $x_0 \asymp F(x_0, y_0)$ and $gy_0 \asymp F(y_0, x_0)$.

We next verify inequality (2.10) of Theorem 2.6. We take $x, y, u, v, z, w \in X$, such that $x \asymp u \asymp z$ and $y \asymp v \asymp w$. Then

$$\begin{aligned}
& G(F(x,y), F(u,v), F(z,w)) \\
= \; & max\Big\{|\frac{1}{6}(x-u) - \frac{1}{4}(y-v)|, |\frac{1}{6}(u-z) - \frac{1}{4}(v-w)|, |\frac{1}{6}(z-x) - \frac{1}{4}(w-y)|\Big\} \\
\leq \; & max\Big\{\frac{1}{4}(|x-u| + |y-v|), \frac{1}{4}(|u-z| + |v-w|), \frac{1}{4}(|z-x| + |w-y|)\Big\} \\
\leq \; & max\Big\{\frac{2}{4}max\{|x-u|, |y-v|\}, \frac{2}{4}max\{|u-z|, v-w|\}, \frac{2}{4}max\{|z-x|, |w-y|\}\Big\} \\
= \; & \frac{2}{4}max\Big\{max\{|x-u|, |u-z|, |z-x|\}, max\{|y-v|, |v-w|, |w-x|\}\Big\} \\
= \; & \frac{1}{2}max\{G(x,u,z), G(y,v,w)\}.
\end{aligned}$$

So condition of (2.10) of Theorem 2.6 is hold with $k = \frac{1}{2}$.

Hence the required condition of Theorem 2.13 satisfied and there exist a unique couple fixed point $(0, 0)$ of F. Note that F does not satisfy the mixed monotone property.

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