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Research Paper

ON THE BASIS PROPERTY OF AN TRIGONOMETRIC FUNCTIONS SYSTEM OF THE FRANKL PROBLEM WITH A NONLOCAL PARITY CONDITION IN THE SOBOLEV SPACE $\overline{W}_p^{2l}(0,\pi)$

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Abstract. In the present paper, we write out the eigenvalues and the corresponding eigenfunctions of the modified Frankl problem with a nonlocal parity condition of the first kind. We analyze the completeness, the basis property, and the minimality of the eigenfunctions in the space $\overline{W}_p^{2l}(0,\pi)$, where $\overline{W}_p^{2l}(0,\pi)$ be the set of functions $f \in W_p^{2l}(0,\pi)$, satisfying of the following conditions: $f^{(2k-1)}(0) = 0, k = 1, 2, ..., l$.

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1. Introduction

The classical Frankl problem was considered in [3]. The problem was further developed in [2, pp.339-345], [8, pp.235-252]. The modified Frankl problem with a nonlocal boundary condition of the first kind was studied in [1, 6]. The basis property of an eigenfunctins of the Frankl problem with a nonlocal parity conditions in the space Sobolev was studied in [7]. In the present paper, we write out the eigenvalues and the corresponding eigenfunctions of the modified Frankl problem with a nonlocal parity condition of the first kind. We analyze the completeness, the basis property, and the minimality of the eigenfunctions in the space $\overline{W}_p^{2l}(0,\pi)$. This analysis may be of interest in itself.

2. Statement of the modified Frankl problem

Definition 2.1. In the domain $D = (D_+ \cup D_{-1} \cup D_{-2})$, we seek a solution of the modified generalized Frankl problem

$$u_{xx} + \operatorname{sgn}(y)u_{yy} + \mu^2 \operatorname{sgn}(x+y)u = 0 \qquad \text{in} \quad (D_+ \cup D_{-1} \cup D_{-2}), \tag{2.1}$$

with the boundary conditions

$$u(1,\theta) = 0, \quad \theta \in [0, \frac{\pi}{2}], \tag{2.2}$$

$$\frac{\partial u}{\partial x}(0,y) = 0, \quad y \in (-1,0) \cup (0,1) \tag{2.3}$$

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$$u(0,y)=u(0,-y), y\in[0,1]. \tag{2.4}$$

where $u(x,y)$ is a regular solution in the class

$$u\in C^0(\overline{D_+\cup D_{-1}\cup D_{-2}})\cap C^2(\overline{D_{-1}})\cap C^2(\overline{D_{-2}}),$$

and where

$$D_+=\{(r,\theta):0<r<1,0<\theta<\frac{\pi}{2}\},$$
$$D_{-1}=\{(x,y):-y<x<y+1,\frac{-1}{2}<y<0\},$$
$$D_{-2}=\{(x,y):x-1<y<-x,0<x<\frac{1}{2}\},$$
$$\kappa\frac{\partial u}{\partial y}(x,+0)=\frac{\partial u}{\partial y}(x,-0),-\infty<\kappa<\infty,0<x<1. \tag{2.5}$$

Theorem 2.2 ([5]). *The eigenvalues and eigenfunctions of problem (1-5) can be written out in two series. In the first series, the eigenvalues $\lambda=\mu_{nk}^2$ are found from the equation*

$$J_{4n}(\mu_{nk})=0, \tag{2.6}$$

where $\mu_{nk}, n=0,1,2,...,k=1,2,...,$ are roots of the Bessel equation (6), $J_\alpha(z)$ is the Bessel function [4], and the eigenfunctions are given by the formula

$$u_{nk}=\begin{cases} A_{nk}J_{4n}(\mu_{nk}r)\cos(4n)(\frac{\pi}{2}-\theta), & in \quad D^+;\\ A_{nk}J_{4n}(\mu_{nk}\rho)\cosh(4n)\psi, & in \quad D_{-1};\\ A_{nk}J_{4n}(\mu_{nk}R)\cosh(4n)\varphi, & in \quad D_{-2},\end{cases} \tag{2.7}$$

where $x=r\cos\theta, y=r\sin\theta$ for $0\le\theta\le\frac{\pi}{2}, 0<r<1,$ and $r^2=x^2+y^2$ in D_+, $x=\rho\cosh\psi, y=\rho\sinh\psi$, for, $0<\rho<1,-\infty<\psi<0,\rho^2=x^2-y^2$, in D_{-1} and $x=R\sinh\varphi, y=-R\cosh\varphi$, for, $0<\varphi<+\infty, R^2=y^2-x^2$, in D_{-2}.

In the second series, the eigenvalues $\tilde{\lambda}=\tilde{\mu}_{nk}^2$ are found from the equation.

$$J_{4(n-\triangle)}(\tilde{\mu}_{nk})=0. \tag{2.8}$$

Where $n=1,2,...,$ and $k=1,2,...,$ and the $(\tilde{\mu}_{nk})$ are the roots of the Bessel equation (8).

$$\tilde{u}_{nk}=\begin{cases} \tilde{A}_{nk}J_{4(n-\triangle)}(\tilde{\mu}_{nk}r)\cos 4(n-\triangle)(\frac{\pi}{2}-\theta), & in \quad D^+;\\ \tilde{A}_{nk}J_{4(n-\triangle)}(\tilde{\mu}_{nk}\rho)[\cosh 4(n-\triangle)\varphi\cos 4(n-\triangle)\frac{\pi}{2} & \\ \quad +\kappa\sinh 4(n-\triangle)\psi\cos 4(n-\triangle)], & in \quad D_{-1};\\ \tilde{A}_{nk}J_{4(n-\triangle)}(\tilde{\mu}_{nk}R)\cosh 4(n-\triangle)\varphi[\cos 4(n-\triangle)\frac{\pi}{2} & \\ \quad -\kappa\sin 4(n-\triangle)\frac{\pi}{2}], & in \quad D_{-2},\end{cases} \tag{2.9}$$

where, $\Delta=\frac{1}{\pi}\arcsin\frac{\kappa}{\sqrt{1+\kappa^2}}, \Delta\in(0,\frac{1}{2})$, and

$$A_{nk}^2\int_0^1 J_{4n}^2(\mu_{nk}r)rdr=1,$$

$$\tilde{A}_{nk}^2\int_0^1 J_{4n-1}^2(\tilde{\mu}_{nk}r)rdr=1,$$

$A_{nk}>0$ and $\tilde{A}_{nk}>0$.

Theorem 2.3. *The function system*

$$\{\cos(4n)(\frac{\pi}{2}-\theta)\}_{n=0}^{\infty}, \{\cos 4(n-\triangle)(\frac{\pi}{2}-\theta)\}_{n=1}^{\infty}, \tag{2.10}$$

is a Riesz basis in $L_2(0,\frac{\pi}{2})$, provided that $\triangle \in (0,\frac{3}{4})$.

Proof. Let us show that any function $f(\theta) \in L_2(0,\frac{\pi}{2})$ can be represented in the form

$$f(\theta) = \sum_{n=0}^{\infty} A_n \cos 4n(\frac{\pi}{2}-\theta) + \sum_{n=1}^{\infty} B_n \cos 4(n-\triangle)(\frac{\pi}{2}-\theta), \tag{2.11}$$

in $L_2(0,\frac{\pi}{4})$. We have

$$\begin{aligned} f(\theta) - f(\frac{\pi}{2}-\theta) &= \sum_{n=1}^{\infty} B_n[\cos 4(n-\triangle)(\frac{\pi}{2}-\theta) - \cos 4(n-\triangle)\theta] \\ &= -2\sin\pi\Delta \sum_{n=1}^{\infty} (-1)^n B_n \sin 4(n-\triangle)(\frac{\pi}{2}-\theta). \end{aligned} \tag{2.12}$$

The function system $\{\sin 4(n-\triangle)(\frac{\pi}{4}-\theta)\}_{n=1}^{\infty}$ is a Riesz basis in $L_2(0,\frac{\pi}{4})$ for $\Delta \in (0,\frac{3}{4})$ (see [5]). Therefore,

$$\sum_{n=1}^{\infty} B_n^2 \le b_1 \|f(\theta) - f(\frac{\pi}{2}-\theta)\|^2_{L_2(0,\frac{\pi}{2})} \le 2b_1\|f\|^2_{L_2(0,\frac{\pi}{4})}. \tag{2.13}$$

And according to the results of [7], we have the estimate

$$\sum_{n=0}^{\infty} A_n^2 + \sum_{n=1}^{\infty} B_n^2 \le b_2\|f\|^2_{L_2(0,\frac{\pi}{2})}. \tag{2.14}$$

By squaring relation (11) and by integrating the resulting relation over the interval $[0,\frac{\pi}{2}]$, we obtain

$$\begin{aligned} \int_0^{\frac{\pi}{2}} f^2(\theta)d\theta &\le 2\int_0^{\frac{\pi}{2}} (\sum_{n=0}^{\infty} A_n \cos 4n(\frac{\pi}{2}-\theta))^2 d\theta + 2\int_0^{\frac{\pi}{2}} F^2(\theta)d\theta \\ &\le c_3(\sum_{n=0}^{\infty} A_n^2 + \sum_{n=1}^{\infty} B_n^2). \end{aligned} \tag{2.15}$$

From inequalities (14) and (15), we obtain the estimate

$$a\|f\|^2_{L_2(0,\frac{\pi}{2})} \le \sum_{n=0}^{\infty} A_n^2 + \sum_{n=1}^{\infty} B_n^2 \le b_3\|f\|^2_{L_2(0,\frac{\pi}{2})}. \tag{2.16}$$

The proof of the theorem is complete. □

3. The completeness, the basis property and minimality of the eigenfunctions

Definition 3.1. Let $\beta < 2-\frac{1}{p}$. Let $\widetilde{W}_p^{2l}(0,\pi)$ be the subspace of the space $W_p^{2l}(0,\pi)$ consisting of functions $f \in W_p^{2l}(0,\pi)$ satisfying the following boundary conditions:

$$f^{2k}(0) = 0, \qquad (k = 0,1,...,l-1) \tag{3.1}$$

and, for $\beta < 1$, let them satisfy condition:

$$\int_0^\pi f^{(2k-1)(\theta)} \widetilde{H}_0^\beta d\theta = 0, \qquad (k = 1, 2, 3, ..., l)$$

where

$$\widetilde{H}_0^\alpha = \frac{\Gamma^2(1-\frac{\alpha}{2})}{\Gamma(1-\alpha)\pi(2\cos\frac{\theta}{2})^\alpha}. \qquad (\alpha = \beta - 2)$$

This restriction on β is connected with applied problems and is natural in this sense.

Definition 3.2. Let $\beta < 2 - \frac{1}{p}$, and let $(\overline{W}_p^{2l}(0, \pi))$ be the set of functions $f \in W_p^{2l}(0, \pi)$ satisfying the following conditions:

$$f^{2k-1}(0) = 0, \qquad (k = 1, ..., l)$$

and, also the following conditions depending on the parameter β : For $\beta < 1$,

$$\int_0^\pi f^{(2k)}(\theta)\widetilde{H}_0^\beta d\theta = 0, \qquad (k = 1, 2, 3, ..., l-1) \tag{3.2}$$

and for $\beta \geq 1$,

$$\int_0^\pi \left(f^{(2k)} - \frac{f^{2l}(-1)^{l-k}}{(1-\frac{\beta}{2})^{2l-2k}} \right) H_0^{\beta-2} d\theta = 0, \qquad (k = 1, 2, 3, ..., l-1) \tag{3.3}$$

$$H_n^\alpha = \frac{2}{\pi(2\cos\frac{\theta}{2})^\alpha} \left\{ \sum_{k=0}^n C_\alpha^k \cos(n-k)\theta - \frac{C_\alpha^n}{2} \right\} \qquad (n \geq 0)$$

and

$$h_n^\beta = \frac{2}{\pi(2\cos\frac{\theta}{2})^\beta} \sum_{k=0}^{n-1} C_\beta^k \sin(n-k)\theta.$$

Remark 3.3. For $\beta = 1$, condition (8) transforms to the condition $f^{2k-2}(\pi) = 0, k = 2, 3, ..., l$ and for $l = 1$ conditions (7) and (8) do not occur.

Theorem 3.4. *The system of function* $\{\cos(n - \frac{\beta}{2})\theta\}_{n=0}^\infty$ *is a Riesz basis in* $(W_p^1(0, \pi))$ *if and only if* $\beta \in (-\frac{1}{p}, 2 - \frac{1}{p}), \beta \neq 1$.

Proof. Using the formula (20) of [7], we have the relation

$$f(\theta) = \sum_{n=1}^\infty B_n \cos(n - \frac{\beta}{2})\theta + B_0, \tag{3.4}$$

where

$$B_n = -\int_0^\pi f'(\theta) h_n^\beta d\theta (n - \frac{\beta}{2})^{-1}. \qquad (n = 1, 2, ...) \tag{3.5}$$

The coefficient B_0, depend on the B_n (see [7]). Consider the formally differentiated series (20):

$$\sum_{n=1}^\infty B_n (n - \frac{\beta}{2}) \sin(n - \frac{\beta}{2})\theta. \tag{3.6}$$

Since the coefficient B_n, are found by formula (21), using the results of [5], we obtain that Series (20) converges to $f'(\theta)$ in the space $L_p(0,\pi)$. Integrating Series (20) from 0 to θ, we obtain the relation

$$f(\theta)-f(0)=\sum_{n=1}^{\infty}B_n\cos(n-\frac{\beta}{2})\theta-\sum_{n=1}^{\infty}B_n. \tag{3.7}$$

Which has a meaning if the following Series converges

$$\sum_{n=1}^{\infty}B_n. \tag{3.8}$$

By using the results of [7], we obtain that the numerical series (24) converges and the relation (23) uniformly converges on $[0,\pi]$, and therefore, it converges in the space $L_p(0,\pi)$. Now we assume that

$$B_0=f(0)-\sum_{n=1}^{\infty}B_n.$$

Then expression (23) coincides with expression (20), and therefore, series (20) converges to function in the space $(W_p^1(0,\pi))$.

Now let us show that the coefficients B_n are uniquely found by using relation (20). Indeed, if series (20) converges in the space $(W_p^1(0,\pi))$, then series (24) converges in the space $L_p(0,\pi)$ (see [7]), this implies that $\lim_{n\to\infty}B_n=0$. For $\beta\in(-\frac{1}{p},2-\frac{1}{p})$. Now let us show that the system $\{\cos(n-\frac{\beta}{2})\theta,1\}_{n=1}^{\infty}$, does not composes a basis for $\beta\notin(-\frac{1}{p},2-\frac{1}{p})$. If $\beta\in(2-\frac{1}{p},4-\frac{1}{p})$ then, using the substitution $\beta-2=\beta'$ and removing the first cosine, we obtain the cosine system $\{\cos(n-\frac{\beta'}{2})\theta_{n=1}^{\infty},1\}$, which as was proved above, composes a basis in $(W_p^1(0,\pi))$, and therefore, the initial cosine system is not minimal in $(W_p^1(0,\pi))$. Analogously, for $\beta\in(-2-\frac{1}{p},-\frac{1}{p})$, the substitution $\beta+2=\beta'$, reduces the initial cosine system to the system with $\beta'\in(-\frac{1}{p},2-\frac{1}{p})$ in which there is no function $(\cos(1-\frac{\beta'}{2})\theta)$, and, therefore the initial cosine system is not complete. Other ranges of the parameter $\beta\in(-\frac{1}{p}+2k,2-\frac{1}{p}+2k),k=\pm1,\pm2,...$ can be considered analogously. Furthermore, for $\beta=2-\frac{1}{p}$ in the space $(W_p^1(0,\pi))$, where $\hat{p}>p$, we have, $-\frac{1}{\hat{p}}<\beta<2-\frac{1}{\hat{p}}$, and therefore, the cosine system composes a basis in $W_{\hat{p}}^1(0,\pi)$, and hence it is complete in the space $(W_p^1(0,\pi))$.

For $\beta=-\frac{1}{p}$, the cosine system is minimal, since as was proved above, the coefficients B_n are found by concrete formulas in the form of an integral. Let us show that for $\beta=2-\frac{1}{p}$, the cosine system is not minimal. By using the results of [5], we obtain that for $\beta=2-\frac{1}{p}$, the cosine system is complete but not minimal, and hence, for $\beta=-\frac{1}{p}$, the cosine system is complete (since it is minimal in this case). Now let us prove that for $\beta=-\frac{1}{p}$, the cosine system does not composes a basis. Let $f(\theta)=\theta$, then $f(\theta)\in(W_p^1(0,\pi)), f'(\theta)=1$, and the coefficients B_n can be calculated by using the formula (21) exactly in the same way as in [5], where it was shown that a series converges to a function not belonging to $L_p(0,\pi)$, thus Theorem 3.4 is proved. □

Theorem 3.5. *Let $p\in(1,\infty),\beta\neq2$. then the cosine system composes a basis in the space $(\overline{W}_p^{2l}(0,\pi))$, if and only if $\beta\in(\frac{-1}{p},2-\frac{1}{p})$, and the expansion of a function $f\in(\overline{W}_p^{2l}(0,\pi))$*

into the series has the form

$$f(\theta)=\sum_{n=1}^{\infty}\widetilde{B}_n\cos(n-\frac{\beta}{2})\theta+\widetilde{B}_0, \tag{3.9}$$

where

$$\widetilde{B}_n=\int_0^{\pi}f^{(2l)}(\theta)H_{n-1}^{\beta-2}(\theta)d\theta(n-\frac{\beta}{2})^{-2l}(-1)^l,\qquad (n=1,2,3,...) \tag{3.10}$$

and

$$\widetilde{B}_0=\int_0^{\pi}f(\theta)H_0^{\beta}d\theta,\qquad \textit{for},\qquad \beta<1,$$

$$\widetilde{B}_0=\int_0^{\pi}\left(f-\frac{f^{2l}(-1)^l}{(1-\frac{\beta}{2})^{2l}}\right)H_0^{\beta-2}d\theta,\qquad \textit{for},\qquad \beta\geq 1.$$

Proof. Let $(\beta\neq 2)$. We first prove the basis properties of the cosine system for $\beta\in(\frac{-1}{p},2-\frac{1}{p})$. Let $f\in((\overline{W}_p^{2l}(0,\pi)))$. The inequality $\beta<2-\frac{1}{p}$ guarantees the existence of integrals (17) and (18). The function $f^{(2l)}$ belongs to the class $L_p(0,\pi)$. Therefore, according to the results of [5], it is possible to write the expansion of the function $f^{(2l)}$ into the following series in cosines:

$$f^{(2l)}(\theta)=\sum_{n=0}^{\infty}(n-\frac{\beta}{2})^{2l}(-1)^l\cos(n-\frac{\beta}{2})\theta. \tag{3.11}$$

Sinces (27) converges in the space $L_p(0,\pi)$ to the function$f^{(2l)}$ for $\beta\in(\frac{-1}{p},2-\frac{1}{p})$. Integrating series (27) from 0 to θ and using (17) for k=l, we obtain that the following series uniformly converges:

$$f^{(2l)}(\theta)=\sum_{n=0}^{\infty}\widetilde{B}_n(n-\frac{\beta}{2})^{2l}(-1)^l\sin(n-\frac{\beta}{2})\theta. \tag{3.12}$$

Now integrating the obtained series from π to θ, we have

$$\begin{aligned} f^{(2l-1)}(\theta)=&\sum_{n=1}^{\infty}\widetilde{B}_n(n-\frac{\beta}{2})^{2l-1}(-1)^{l-1}\cos(n-\frac{\beta}{2})\theta\\ &-\sum_{n=1}^{\infty}\widetilde{B}_n(n-\frac{\beta}{2})^{2l-1}(-1)^{l-1}\cos(n-\frac{\beta}{2})\pi+f^{(2l-1)}(\pi). \end{aligned} \tag{3.13}$$

According to Corollary 2 of [5]. We have

$$\parallel H_{n-1}^{\beta-2}\parallel\leq c,\qquad (n\geq 1,\frac{1}{q}+\frac{1}{p}=1),$$

therefore, applying (26) and the Holder inequality, we have $|\ \widetilde{B}_n(n-\frac{\beta}{2})^{2L+1}\ |\leq\parallel f^{2L}\parallel_{L^p}\parallel H_{n-1}^{\beta-2}\parallel_{L_q}\leq\text{const},n\geq 1.$

The obtained estimates immediately imply that the numerical series in converges ,therefore,the functional series in (29) also converges. Now, let $\beta<1$. Multiplying (29) by $\widetilde{H}_0^{\beta}(\theta)$, integrating the obtained relation in the limits from 0 to π, and by using the results of [9], we obtain

$$f^{(2l-1)}(\theta) = \sum_{n=1}^{\infty} \widetilde{B}_n (n - \frac{\beta}{2})^{2l-1} (-1)^{l-1} \cos(n - \frac{\beta}{2})\theta. \tag{3.14}$$

Now, let $\beta \geq 1$. In this case, we multiply series (29) by $\widetilde{H}_0^{\beta-2}(\theta)$, integrate from 0 to π, we immediately obtain the required relation (29). Analogously it is proved that if we differentiate series (20) k times, where $k = 0, 1, 2, ..., 2l-1$, then the obtained series uniformly converges to $f^{(k)}(\theta)$ on $[0,\pi]$. Therefore, the cosine system composes a basis in the space $(\overline{W}_p^{2l}(0,\pi))$ for $\beta \in (\frac{-1}{p}, 2 - \frac{1}{p})$. For $\beta < (\frac{-1}{p})$, the cosine system is not complete in $L_p(0,\pi)$, according to [5]. Therefore, series (28), cannot approximate an arbitrary function $f^{(2l)}(\theta) \in L_p(0,\pi)$. Hence for $\beta < (1 - \frac{1}{p})$, the cosine system is not complete in the space $(\overline{W}_p^{2l}(0,\pi))$. For $\beta = (1 - \frac{1}{p})$, the cosine system is complete and minimal in the space $(\overline{W}_p^{2l}(0,\pi))$.

For $p = 2$, the cosine system composes a Riesz basis. the proof of Theorem 3.5 is complete. □

Remark 3.6. Let $\triangle \in (-\infty, +\infty)$ the system of function (10) a Riesz basis in $(\overline{W}_p^{2l}(0,\pi))$, if and only if $\triangle \in (\frac{-1}{4}, 0) \cup (0, \frac{3}{4})$.

If $\triangle \geq \frac{3}{4}, \triangle \neq 1, 2, 3, ...$, then system (10) is complete but is not minimal in $(\overline{W}_p^{2l}(0,\pi))$.

If $\triangle = \frac{-1}{4}$, then system (10) is complete and minimal but is not basis in $(\overline{W}_p^{2l}(0,\pi))$.

If $\triangle < \frac{-1}{4}, \triangle \neq 1, 2, 3, ...$, then system (10) is not complete but is minimal and Riesz basis in $(\overline{W}_p^{2l}(0,\pi))$.

Proof. The proof of Remark 3.6 reproduces that of Theorem 2.3 and Theorem 3.5. □

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