



EXISTENCE OF THREE SOLUTIONS FOR HEMIVARIATIONAL
INEQUALITIES DRIVEN WITH IMPULSIVE EFFECTS

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ABSTRACT. In this paper we prove the existence of at least three solutions to the following second-order impulsive system:

$$\begin{cases} -(\rho(x)\dot{u})' + A(x)u \in \lambda(\partial j(x, u(x)) + \mu\partial k(x, u(x))), & \text{a.e. } t \in (0, T), \\ \Delta(\rho(x)\dot{u}^i(x_j)) = \rho(x_j^+)\dot{u}^i(x_j^+) - \rho(x_j^-)\dot{u}^i(x_j^-) = I_{ij}(u^i(x_j)), \\ i = 1, \dots, N, \quad j = 1, \dots, l, \\ \alpha_1\dot{u}(0) - \alpha_2u(0) = 0, \quad \beta_1\dot{u}(T) + \beta_2u(T) = 0, \end{cases}$$

where $A : [0, T] \rightarrow \mathbb{R}^{N \times N}$ is a continuous map from the interval $[0, T]$ to the set of N -order symmetric matrixes. The approach is fully based on a recent three critical points theorem of Teng [K. Teng, Two nontrivial solutions for hemivariational inequalities driven by nonlocal elliptic operators, *Nonlinear Anal. (RWA)* 14 (2013) 867-874].

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1. Introduction

The aim of this paper is to establish the existence of at least three solutions for the following second-order impulsive system

$$(1.1) \quad \begin{cases} -(\rho(x)\dot{u})' + A(x)u \in \lambda(\partial j(x, u(x)) + \mu\partial k(x, u(x))), & \text{a.e. } t \in [0, T], \\ \Delta(\rho(x)\dot{u}^i(x_j)) = \rho(x_j^+)\dot{u}^i(x_j^+) - \rho(x_j^-)\dot{u}^i(x_j^-) = I_{ij}(u^i(x_j)), \\ i = 1, \dots, N, \quad j = 1, \dots, l, \\ \alpha_1\dot{u}(0) - \alpha_2u(0) = 0, \quad \beta_1\dot{u}(T) + \beta_2u(T) = 0, \end{cases}$$

where $A : [0, T] \rightarrow \mathbb{R}^{N \times N}$ is a continuous map from the interval $[0, T]$ to the set of N -order symmetric matrices, $\rho \in L^\infty[0, T]$ with $\text{ess inf}_{[0, T]} \rho > 0$, $0 < \rho(0), \rho(T) < +\infty$, $\alpha_1, \alpha_2, \beta_1, \beta_2$ are positive constants, $\lambda, \mu \in \mathbb{R}$ are two parameters, $u(x) = (u^1(x), \dots, u^N(x))$, $x_j, j = 1, \dots, l$, are the instants where the impulses occur and $0 = x_0 < x_1 < x_2 < \dots < x_l < x_{l+1} = T$, $I_{ij} : \mathbb{R} \rightarrow \mathbb{R}$ ($i = 1, \dots, N, j = 1, \dots, l$) are continuous and $j, k : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ are a measurable function such that for all $x \in [0, T]$, $j(x, \cdot), k(x, \cdot)$ are locally Lipschitz and $\partial j(x, \cdot), \partial k(x, \cdot)$ denotes the generalized subdifferential in the sense of Clarke [1].

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In recent years, a great deal of work has been done in the study of the existence of solutions for impulsive boundary value problems, by which a number of chemotherapy, population dynamics, optimal control, ecology, industrial robotics and physics phenomena are described. For the background, theory and applications of impulsive differential equations, we refer interested readers to [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21] and references therein. For example, in [16], Nieto and O Regan studied the nonlinear Dirichlet impulsive problem:

$$(1.2) \quad \begin{aligned} -u''(t) + u(t) &= f(t, u(t)), & \text{a.e. } t \in [0, T] \setminus \{t_1, \dots, t_p\}, \\ \Delta u'(t_j) &= I_j(u(t_j)), & j = 1, \dots, p, \end{aligned}$$

$$(1.3) \quad u(0) = u(T) = 0,$$

where $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$, $I_j \in C(\mathbb{R}, \mathbb{R})$ and $j = 1, \dots, p$. By the least action principle, the existence of a solution was obtained by assuming sublinear growth on the nonlinearity and impulses.

In [9] Zhang and Dai discussed the problem:

$$-u''(t) + g(t)u(t) = f(t, u(t)), \quad \text{a.e. } t \in [0, T] \setminus \{t_1, \dots, t_p\},$$

with impulse conditions (1.2), and Dirichlet boundary condition (1.3), where $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$, $g \in L^\infty([0, T], \mathbb{R})$. By using variational methods, the existence result of an infinite number of solutions was obtained.

Tian and Ge in [20] studied the existence of multiple solutions for the following equation with impulsive effect

$$\begin{cases} -(p(t)u')' + q(t)u = f(t, u(t)), & \text{a.e. } t \in [0, 1] \setminus \{t_1, \dots, t_p\}, \\ \Delta(p(x)u'(t_j)) = I_j(u(t_j)), & j = 1, \dots, p, \\ \alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \sigma u'(1) = 0. \end{cases}$$

By applying variational methods and upper and lower solutions methods, they obtained the existence of at least four solutions and gave some accurate characteristics of the solutions.

The qualitative analysis of solutions of differential inclusion has been attracting a lot of interest, because the differential inclusions are playing an increasingly important role in control systems, mechanical systems, economical systems, game theory, and biological systems, see for instance [22, 23, 24, 25, 26, 27] and the references therein.

Inspired by the above articles, in this paper, by variational methods, we would like to investigate the existence of solutions to problem (1.1).

The paper is organized as follows. In Section 2, we give preliminary facts and provide some basic properties which are needed later. Section 3 is devoted to our results on existence of three solutions.

2. Preliminaries

In this section, we present some preliminaries and lemmas that are useful to the proof of the main results. For the convenience of the reader, we also present here the necessary definitions.

We assume that A satisfies the following conditions:

(F1) $A(x) = (d_{ij}(x))$ is a symmetric continuous matrix with $d_{ij} \in L^\infty([0, T])$ for every $x \in [0, T]$.

(F2) There exists a positive constant ζ such that $A(x)u \cdot u \geq \zeta|u|^2$ for every $u \in \mathbb{R}^N$ and a.e. in $[0, T]$.

We introduce some functional spaces. We use $|\cdot|$ to denote the Euclidean norm in \mathbb{R}^N and H_T^1 the Sobolev space

$$H_T^1 := \{u \in L^2([0, T]; \mathbb{R}^N) : \dot{u} \in L^2([0, T]; \mathbb{R}^N)\}$$

where \dot{u} is weak derivative of u , with the inner product

$$\langle u, v \rangle = \int_0^T (u(x), v(x))dx + \int_0^T (\rho(x)\dot{u}(x), \dot{v}(x))dx,$$

where (\cdot, \cdot) denotes the inner product in \mathbb{R}^N . The corresponding norm is defined by

$$(2.1) \quad \|u\|_{H_T^1} = \left(\int_0^T |u(x)|^2 dx + \int_0^T \rho(x)|\dot{u}(x)|^2 dx \right)^{\frac{1}{2}}, \quad u \in H_T^1,$$

which is equivalent to the usual one. For every $u, v \in H_T^1$, we also define

$$(2.2) \quad \langle u, v \rangle_1 = \int_0^T (A(x)u(x), v(x))dx + \int_0^T (\rho(x)\dot{u}(x), \dot{v}(x))dx,$$

and observe that, by the assumptions (F1) and (F2), (2.2) defines an inner product in H_T^1 , whose corresponding norm is

$$(2.3) \quad \|u\| = \langle u, u \rangle_1^{\frac{1}{2}}.$$

It is clear that H_T^1 is a reflexive Banach space. A simple computation shows that

$$(2.4) \quad (A(x)u, u) = \sum_{i,j=1}^N d_{i,j}(x)u^i u^j \leq \sum_{i,j=1}^N \|d_{i,j}\|_{\infty} |u|^2,$$

for every $x \in [0, T]$ and $u \in \mathbb{R}^N$, thus, putting together (A2) and (2.4), we have

$$(2.5) \quad \sqrt{m_0} \|u\|_{H_T^1} \leq \|u\| \leq \sqrt{M_0} \|u\|_{H_T^1},$$

where $m_0 = \min\{1, \zeta\}$ and $M_0 = \max\{1, \sum_{i,j=1}^N \|a_{i,j}\|_{\infty}\}$, that is the norm $\|\cdot\|$ is equivalent to (2.1).

Lemma 2.1. *For $u \in H_T^1$, we have $\|u\|_{\infty} \leq \gamma_0 \|u\|_{H_T^1}$, where*

$$\gamma_0 = \sqrt{2} \max \left\{ \frac{1}{\sqrt{T}}, \frac{\sqrt{T}}{(\text{ess inf}_{[0,T]} \rho)^{\frac{1}{2}}} \right\}.$$

Proof. For $u \in H_T^1$, by this fact $|u| \in \mathbb{R}$ and mean-value theorem, one can get

$$|u(\theta)| = \frac{1}{T} \int_0^T |u(\tau)| d\tau,$$

for some $\theta \in [0, T]$. Thus, for $x \in [0, T]$, using Hölder's inequality,

$$\begin{aligned}
|u(x)| &= \left| u(\theta) + \int_{\theta}^x \dot{u}(\tau) d\tau \right| \\
&\leq |u(\theta)| + \int_{\theta}^x |\dot{u}(\tau)| d\tau \\
&\leq \frac{1}{T} \int_0^T |u(\tau)| d\tau + \int_0^T |\dot{u}(\tau)| d\tau \\
&\leq T^{-\frac{1}{2}} \left(\int_0^T |u(\tau)|^2 d\tau \right)^{\frac{1}{2}} + T^{\frac{1}{2}} \left(\int_0^T |\dot{u}(\tau)|^2 d\tau \right)^{\frac{1}{2}} \\
&\leq T^{-\frac{1}{2}} \left(\int_0^T |u(\tau)|^2 d\tau \right)^{\frac{1}{2}} + \frac{T^{\frac{1}{2}}}{(\text{ess inf}_{[0,T]} \rho)^{\frac{1}{2}}} \left(\int_0^T \rho(\tau) |\dot{u}(\tau)|^2 d\tau \right)^{\frac{1}{2}} \\
&\leq \sqrt{2} \max \left\{ T^{-\frac{1}{2}}, \frac{T^{\frac{1}{2}}}{(\text{ess inf}_{[0,T]} \rho)^{\frac{1}{2}}} \right\} \|u\|_{H_T^1},
\end{aligned}$$

which completes the proof. \square

Now by using (2.5) and Lemma 2.1, there exist a positive constant k such that

$$(2.6) \quad \|u\|_{\infty} \leq M_1 \|u\|,$$

where $M_1 = \frac{1}{\sqrt{m_0}} \gamma_0$.

Also, we present some preliminaries, basic notions and results of nonsmooth analysis, namely, the calculus for locally Lipschitz functionals developed by Clarke [1], that are useful to the proof to the main results.

Let $(X, \|\cdot\|_X)$ be a Banach space, $(X^*, \|\cdot\|_{X^*})$ be its topological dual, and $\varphi : X \rightarrow \mathbb{R}$ be a functional. We recall that φ is locally Lipschitz if, for all $u \in X$, there exist a neighborhood U of u and a real number $L_U > 0$ such that

$$|\varphi(x) - \varphi(y)| \leq L_U \|x - y\|_X, \quad \forall x, y \in U.$$

If f is locally Lipschitz and $u \in X$, the generalized directional derivative of φ at u along the direction $v \in X$ is

$$\varphi^{\circ}(u; h) = \limsup_{w \rightarrow u, t \downarrow 0^+} \frac{\varphi(w + th) - \varphi(w)}{t}.$$

The generalized gradient of φ at u is the set

$$\partial\varphi(u) = \{u^* \in X^* : \langle u^*, v \rangle \leq \varphi^{\circ}(u; v) \text{ for all } v \in X\}.$$

So $\partial\varphi : X \rightarrow 2^{X^*}$ is a multifunction. The function $(u, v) \mapsto \varphi^{\circ}(u; v)$ is upper semicontinuous and

$$\varphi^{\circ}(u; v) = \max\{\langle \xi, v \rangle : \xi \in \partial\varphi(u)\} \quad \text{for all } v \in X.$$

We say that φ has compact gradient if $\partial\varphi$ maps bounded subsets of X into relatively compact subsets of X^* .

We say that $u \in X$ is a critical point of locally Lipschitz functional φ if $0 \in \partial\varphi(u)$.

In the proof of our main results, we shall use nonsmooth critical point theory. For this, we first present an important definition.

Definition 2.1. *An operator $A : X \rightarrow X^*$ is of type $(S)_+$ if, for any sequence $\{u_n\}$ in X , $u_n \rightharpoonup u$ and $\limsup_{n \rightarrow +\infty} \langle A(u_n), u_n - u \rangle \leq 0$ imply $u_n \rightarrow u$.*

Definition 2.2. *A locally Lipschitz function $\varphi : X \rightarrow \mathbb{R}$ satisfies the nonsmooth Palais-Smale condition at level $c \in \mathbb{R}$ (nonsmooth $(PS)_c$ -condition for short) if any sequence $\{u_n\}_{n \geq 1} \subseteq X$ which satisfies*

(2.1) $J(u_n) \rightarrow c$;

(2.2) *there exist $\{\epsilon_n\} \subset \mathbb{R}$, $\epsilon_n \downarrow 0$ such that $J^\circ(u_n; v - u_n) + \epsilon_n \|v - u_n\|_X \geq 0$, for all $v \in X$ and all $n \in \mathbb{N}$;*

admits a strongly convergent subsequence.

If this is true for every $c \in \mathbb{R}$, we say that J satisfies the non smooth (PS)-condition.

Lemma 2.2. (*[28], Proposition 1.1*). *Let $\varphi \in C^1(X)$ be a functional. Then φ is locally Lipschitz and*

$$\begin{aligned} \varphi^\circ(u; v) &= \langle \varphi'(u), v \rangle, \quad \forall u, v \in X, \\ \partial\varphi(u) &= \{\varphi'(u)\}, \quad \forall u \in X, \end{aligned}$$

where φ' is the Frechet derivative of φ .

Lemma 2.3. (*[29], Lemma 6*). *Let $\varphi : X \rightarrow \mathbb{R}$ be a locally Lipschitz functional with a compact gradient. Then φ is sequentially weakly continuous.*

We say that $u \in X$ is a critical point of locally Lipschitz functional φ if $0 \in \partial\varphi(u)$.

In the proof of our main results, we shall use Theorem 2.1. For this, we first present an important definition.

Definition 2.3. *Let $\Phi : X \rightarrow \mathbb{R}$ be a locally Lipschitz functional and $\Psi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semi continuous functional whose restriction to the set $\text{dom}(\Psi) = \{u \in X : \Psi(u) < \infty\}$ is continuous. Then, $\Phi + \Psi$ is a Motreanu-Panagiotopoulos functional.*

Definition 2.4. *Let $\Phi + \Psi$ be a Motreanu-Panagiotopoulos functional, $u \in X$. Then u is a critical point of $\Phi + \Psi$ if for every $v \in X$, $\Phi^0(u; v - u) + \Psi(v) - \Psi(u) \geq 0$.*

The following lemma one basic properties of the generalized gradients:

Lemma 2.4. *Let $\varphi_1, \varphi_2 : X \rightarrow \mathbb{R}$ be a locally Lipschitz functionals. Then, for every $u, v \in X$, the following conditions hold:*

(6.1) $\partial\varphi_1(u)$ is convex and weakly* compact;

(6.2) the set-value mapping $\partial\varphi_1 : X \rightarrow 2^{X^*}$ is weakly* upper semicontinuous;

(6.3) $\varphi_1^\circ(u; v) = \max_{u^* \in \partial\varphi_1(u)} \langle u^*, v \rangle \leq L_U \|v\|$, with L_U as in definition of locally Lipschitz functionals;

(6.4) $\partial(\lambda\varphi_1)(u) = \lambda\partial\varphi_1(u)$ for every $\lambda \in \mathbb{R}$;

(6.5) $\partial(\varphi_1 + \varphi_2)(u) \subseteq \partial\varphi_1(u) + \partial\varphi_2(u)$ for every $\lambda \in \mathbb{R}$;

The goal of this work is to establish some new criteria for system (1.1) to have at least three weak solutions in X , by means of a very recent abstract critical points result of Teng [30]. First, we recall the following result of ([30], Theorem 2.4), with easy manipulations, that we are going to use in the sequel.

Theorem 2.1. *Let X be a reflexive real Banach space, let Ψ be a convex, proper, lower semicontinuous functional and $\Phi : X \rightarrow \mathbb{R}$ be a locally Lipschitz functional with compact gradient $\partial\Phi$ and Φ is nonconstant. Suppose that*

- (A1) $\Theta : X \rightarrow \mathbb{R}$ is a locally Lipschitz functional with compact gradient $\partial\Theta$;
- (A2) there exists an interval $\Lambda \subset \mathbb{R}$ and a number $\eta > 0$, such that for every $\lambda \in \Lambda$ and every $\mu \in [-\eta, \eta]$ the functional $J_{\lambda, \mu} = \Psi + \lambda(\Phi + \mu\Theta)$ is coercive in X ;
- (A3) The functional $J_{\lambda, \mu}$ satisfies the Palais-Smale condition for every $\lambda \in \Lambda$ and every $\mu \in [-\eta, \eta]$;
- (A4) There exists $r \in (\inf_{u \in X} \Phi(u), \sup_{u \in X} \Phi(u))$ such that the following two numbers

$$\varphi_1(r) = \inf_{u \in \Phi^{-1}(I_r)} \frac{\inf_{v \in \Phi^{-1}(r)} \Psi(v) - \Psi(u)}{\Phi(u) - r}$$

$$\text{and } \varphi_2(r) = \sup_{u \in \Phi^{-1}(I^r)} \frac{\inf_{v \in \Phi^{-1}(r)} \Psi(v) - \Psi(u)}{\Phi(u) - r}$$

satisfy $\varphi_1(r) < \varphi_2(r)$, where $I_r = (-\infty, r)$ and $I^r = (r, +\infty)$.

If $(\varphi_1(r), \varphi_2(r)) \cap \Lambda \neq \emptyset$, then for every compact interval $[a, b] \subset (\varphi_1(r), \varphi_2(r)) \cap \Lambda$, there exists $\delta \in (0, \eta)$ such that if $|\mu| < \delta$, the functional $J_{\lambda, \mu}$ admits at least three critical points for every $\lambda \in [a, b]$.

The functional $J_{\lambda, \mu} : X \rightarrow \mathbb{R}$ corresponding to problem (1.1) is defined by

$$(2.7) \quad \begin{aligned} J_{\lambda, \mu}(u) &= \frac{1}{2} \|u\|^2 - \sum_{j=1}^m \sum_{i=1}^N \int_0^{u^i(x_j)} I_{ij}(s) ds + \frac{\alpha_2 \rho(0)}{2\alpha_1} |u(0)|^2 \\ &+ \frac{\beta_2 \rho(T)}{2\beta_1} |u(T)|^2 - \lambda \left[\int_0^T j(x, u(x)) dx + \mu \int_0^T k(x, u(x)) dx \right]. \end{aligned}$$

In order to study problem (1.1), we will use the functionals $\Phi, \Psi : H_T^1 \rightarrow \mathbb{R}$ define by

$$(2.8) \quad \begin{aligned} \Psi(u) &= \frac{1}{2} \|u\|^2 - \sum_{j=1}^m \sum_{i=1}^N \int_0^{u^i(x_j)} I_{ij}(s) ds + \frac{\alpha_2 \rho(0)}{2\alpha_1} |u(0)|^2 + \frac{\beta_2 \rho(T)}{2\beta_1} |u(T)|^2 \\ &= \phi(u) + \psi(u), \end{aligned}$$

$$(2.9) \quad \Phi(u) = - \int_0^T j(x, u(x)) dx, \quad \Theta(u) = - \int_0^T k(x, u(x)) dx.$$

Now, we will establish the variational principle for problem (1.1). For this purpose our hypotheses on the nonsmooth potential $F(x, u)$ and real continuous function I_{ij} are the following:

- (H1) For all $u \in \mathbb{R}$, the function $x \rightarrow j(x, u)$ is measurable;
- (H2) For all $x \in [0, T]$, the function $u \rightarrow j(x, u)$ is locally Lipschitz and $j(x, 0) = 0$;
- (H3) There exist $a, b \in L^1([0, T]; \mathbb{R})$ and $1 \leq r < 2$ such that $|u^*| \leq a(x) + b(x)|u|^{r-1}$ for all $x \in [0, T]$, $x \in \mathbb{R}$ and $u^* \in \partial j(x, u)$;

(I1) There exist constants $a_{ij}, b_{ij} > 0$ and $\gamma_{ij} \in [0, 1)$, $j = 1, 2, \dots, m$, $i = 1, 2, \dots, N$ such that $|I_{ij}(x)| \leq a_{ij} + b_{ij}|x|^{\gamma_{ij}}$ for all $x \in \mathbb{R}$, $j = 1, 2, \dots, m$, $i = 1, 2, \dots, N$.

Definition 2.5. We say that $u \in X$ is a weak solution to problem (1.1) if

$$\begin{aligned} & \int_0^T [(\rho(x)\dot{u}(x), \dot{v}(x)) + (A(x)u(x), v(x)) - (\lambda(u^*(x) + \mu v^*(x)), v(x))] dx \\ & + \frac{\alpha_2 \rho(0)}{\alpha_1} (u(0), v(0)) + \frac{\beta_2 \rho(T)}{\beta_1} (u(T), v(T)) - \sum_{j=1}^m \sum_{i=1}^N I_{ij}(u^i(x_j)) v^i(x_j) = 0, \end{aligned}$$

for all $u^* \in \partial j(x, u(x))$, $v^* \in \partial k(x, u(x))$, $v \in X$, for a.e. $x \in [0, T]$.

Proposition 2.1. Assume that $j(x, u)$ and $k(x, u)$ satisfy the hypotheses (H1)-(H3), the functional $J_{\lambda, \mu} : X \rightarrow \mathbb{R}$ is well defined and locally Lipschitz on X . Moreover, every critical point $u \in X$ of $J_{\lambda, \mu}$ is a solution of problem (1.1).

proof. The proof is similar to [[31], Lemmas 3.5 and 4.4] and is omitted. \square

According to Proposition 2.1, we know that in order to find solutions of problem (1.1), it suffices to obtain the critical points of the functional $J_{\lambda, \mu}$.

3. Main result

In this section we present our main results. Now, we will apply Theorem 2.1 to obtain some existence and multiplicity results for problem (1.1).

Before given the our main result, we need the following lemmas.

Lemma 3.1. Assume that A satisfy the hypotheses (F1)-(F2), $j(x, u)$ and $k(x, u)$ satisfy the hypotheses (H1)-(H3) and I_{ij} satisfies the hypotheses (I1), then the functional $J_{\lambda, \mu} : X \rightarrow \mathbb{R}$ is coercive for every $\lambda, \mu \in \mathbb{R}$.

proof. By condition (I1), when $u^i(x_j) < 0$, we get

$$\begin{aligned} a_{ij} u^i(x_j) + \frac{b_{ij}(-1)^{\gamma_{ij}}}{\gamma_{ij} + 1} (u^i(x_j))^{\gamma_{ij} + 1} & \leq \int_{u^i(x_j)}^0 I_{ij}(s) ds \\ & \leq -a_{ij} u^i(x_j) - \frac{b_{ij}(-1)^{\gamma_{ij}}}{\gamma_{ij} + 1} (u^i(x_j))^{\gamma_{ij} + 1}, \end{aligned}$$

when $u^i(x_j) \geq 0$, we have

$$\begin{aligned} -a_{ij} u^i(x_j) - \frac{b_{ij}}{\gamma_{ij} + 1} (u^i(x_j))^{\gamma_{ij} + 1} & \leq \int_0^{u^i(x_j)} I_{ij}(s) ds \\ & \leq a_{ij} u^i(x_j) + \frac{b_{ij}}{\gamma_{ij} + 1} (u^i(x_j))^{\gamma_{ij} + 1}. \end{aligned}$$

Thus,

$$(3.1) \quad \left| \int_0^{u^i(x_j)} I_{ij}(s) ds \right| \leq a_{ij} |u^i(x_j)| + \frac{b_{ij}}{\gamma_{ij} + 1} |u^i(x_j)|^{\gamma_{ij}+1} \\ \leq a_{ij} |u(x_j)| + \frac{b_{ij}}{\gamma_{ij} + 1} |u(x_j)|^{\gamma_{ij}+1}.$$

Therefore, by (2.6), one can get

$$(3.2) \quad \left| \sum_{j=1}^m \sum_{i=1}^N \int_0^{u^i(x_j)} I_{ij}(s) ds \right| \leq \sum_{j=1}^m \sum_{i=1}^N \left(a_{ij} M_1 \|u\| + \frac{b_{ij}}{\gamma_{ij} + 1} M_1^{\gamma_{ij}+1} \|u\|^{\gamma_{ij}+1} \right).$$

Also, by (H2), (H3) and the Lebourg's mean value theorem, we have

$$(3.3) \quad |j(x, u)| = |j(x, u) - j(x, 0)| = |(u^*, u)| \leq a(x)|u| + b(x)|u|^r \\ |k(x, u)| = |k(x, u) - k(x, 0)| = |(u^*, u)| \leq a(x)|u| + b(x)|u|^r$$

for all $u \in \mathbb{R}^N$ and $x \in [0, T]$. Thus, by (2.6) and (2.7), (3.2) and (3.3), one can get

$$J_{\lambda, \mu}(u) = \frac{1}{2} \|u\|^2 - \sum_{j=1}^m \sum_{i=1}^N \int_0^{u^i(x_j)} I_{ij}(s) ds + \frac{\alpha_2 \rho(0)}{2\alpha_1} |u(0)|^2 \\ + \frac{\beta_2 \rho(T)}{2\beta_1} |u(T)|^2 - \lambda \left[\int_0^T j(x, u(x)) dx + \mu \int_0^T k(x, u(x)) dx \right] \\ \geq \frac{1}{2} \|u\|^2 - \sum_{j=1}^m \sum_{i=1}^N \left(a_{ij} M_1 \|u\| + \frac{b_{ij}}{\gamma_{ij} + 1} M_1^{\gamma_{ij}+1} \|u\|^{\gamma_{ij}+1} \right) \\ + \frac{\alpha_2 \rho(0)}{2\alpha_1} |u(0)|^2 + \frac{\beta_2 \rho(T)}{2\beta_1} |u(T)|^2 \\ - |\lambda|(1 + |\mu|) \left[\|u\|_\infty \int_0^T a(x) dx + \|u\|_\infty^r \int_0^T b(x) dx \right] \\ \geq \frac{1}{2} \|u\|^2 - \sum_{j=1}^m \sum_{i=1}^N \left(a_{ij} M_1 \|u\| + \frac{b_{ij}}{\gamma_{ij} + 1} M_1^{\gamma_{ij}+1} \|u\|^{\gamma_{ij}+1} \right) \\ - |\lambda|(1 + |\mu|) \left[M_1 \|u\| \int_0^T a(x) dx + M_1^r \|u\|^r \int_0^T b(x) dx \right].$$

Since $1 \leq r < 2$, then $J_{\lambda, \mu}$ is coercive for every $\lambda, \mu \in \mathbb{R}$. \square

Lemma 3.2. *Assume that A satisfy the hypotheses (F1)-(F2), $j(x, u)$ satisfies the hypotheses (H1)-(H3). Then, the functional $\Phi : X \rightarrow \mathbb{R}$ is a locally Lipschitz functional with compact gradient.*

proof. Clearly, Φ is locally Lipschitz on H_T^1 . Now we shall show that the set-valued function $\partial\Phi : H_T^1 \rightarrow 2^{(H_T^1)^*}$ is compact. To this end, let us fix a bounded sequence $\{u_n = (u_n^1, \dots, u_n^N)\} \subset H_T^1$ and $u_n^* \in \partial\Phi(u_n)$ for all $n \in \mathbb{N}$ such that $\langle u_n^*, v \rangle = \int_0^T (u_n^*(x), v(x)) dx$ for

every $v \in H_T^1$. Let $L > 0$ be a Lipschitz constant for Φ , restricted to a bounded set where the sequence $\{u_n\}$ lies, then $\|u_n^*\|_{(H_T^1)^*} \leq L$ for all $n \in \mathbb{N}$. Up to a subsequence, $\{u_n^*\}$ weakly converges to some u^* in $(H_T^1)^*$. We shall show that the convergence is strong. Assume to the contrary, that is, we assume there exists $\epsilon > 0$ such that $\|u_n^* - u^*\|_{(H_T^1)^*} > \epsilon$ for all $n \in \mathbb{N}$. Hence for all $n \in \mathbb{N}$, there exists $v_n \in B^N(0, 1)(B^N(0, 1) = \{u = (u^1, \dots, u^N) \in H_T^1 : \|u\| \leq 1\})$ such that

$$(3.4) \quad \langle u_n^* - u^*, v_n \rangle > \epsilon.$$

Since $\{v_n\}$ is bounded in H_T^1 , then up to a subsequence, there is a $v \in H_T^1$ such that $v_n \rightharpoonup v$ in H_T^1 and $v_n \rightarrow v$ in $C([0, T])$, $v_n \rightarrow v$ in $L^q[0, T]$ ($1 \leq q \leq 2$). From (H3), one can get

$$\begin{aligned} \langle u_n^* - u^*, v_n \rangle &= \langle u_n^*, v_n - v \rangle + \langle u_n^* - u^*, v \rangle + \langle u^*, v - v_n \rangle \\ &\leq C_1(\|v_n - v\|_{L^1} + \|v_n - v\|_{L^q}) + \langle u_n^* - u^*, v \rangle + \langle u^*, v - v_n \rangle \rightarrow 0, \end{aligned}$$

as $n \rightarrow +\infty$, which contradicts (3.4). \square

Lemma 3.3. *The functional $\Psi : X \rightarrow \mathbb{R}$ is a sequentially weakly lower semicontinuous.*

proof. Let $\{u_n\}$ be a weakly convergent sequence to u in H_1^T , then $\|u\| \leq \liminf_{n \rightarrow \infty} \|u_n\|$. We have that $\{u_n\}$ converges uniformly to u on $C[0, T]$. Then we get

$$\begin{aligned} \liminf_{n \rightarrow \infty} \Psi(u_n) &= \liminf_{n \rightarrow \infty} \left[\frac{1}{2} \|u_n\|^2 - \sum_{j=1}^m \sum_{i=1}^N \int_0^{u_n^i(x_j)} I_{ij}(s) ds + \frac{\alpha_2 \rho(0)}{2\alpha_1} |u_n(0)|^2 + \frac{\beta_2 \rho(T)}{2\beta_1} |u_n(T)|^2 \right] \\ &\geq \frac{1}{2} \|u\|^2 - \sum_{j=1}^m \sum_{i=1}^N \int_0^{u^i(x_j)} I_{ij}(s) ds + \frac{\alpha_2 \rho(0)}{2\alpha_1} |u(0)|^2 + \frac{\beta_2 \rho(T)}{2\beta_1} |u(T)|^2. \end{aligned}$$

Hence, $\Psi : X \rightarrow \mathbb{R}$ is a sequentially weakly lower semicontinuous. \square

Therefore, we have the following remark.

Remark 3.1. *By the Definition 2.3, the functional $J_{\lambda, \mu}$ is of a Motreanu-Panagiotopoulos functional on X .*

Lemma 3.4. *Assume that A satisfy the hypotheses (F1)-(F2), $j(x, u)$ and $k(x, u)$ satisfy the hypotheses (H1)-(H3) and I_{ij} satisfies the hypotheses (I1). Then, the functional $J_{\lambda, \mu}$ satisfies the (PS)-condition for every $\lambda, \mu \in \mathbb{R}$.*

proof. Let $\{u_n\}$ be a sequence in X such that $J_{\lambda, \mu}(u_n)$ is bounded and

$$J_{\lambda, \mu}^\circ(u_n; v - u_n) + \epsilon_n \|v - u_n\| \geq 0,$$

for every $v \in X$, where $\epsilon_n \rightarrow 0^+$ as $n \rightarrow \infty$. For $v = u$, by Definition 2.2 and Lemma 2.4, we get

$$\begin{aligned}
0 &\leq \epsilon_n \|u - u_n\| + J_{\lambda, \mu}^{\circ}(u_n; u - u_n) \\
&= \epsilon_n \|u - u_n\| + (\Psi + \lambda(\Phi + \mu\Theta))^{\circ}(u_n; u - u_n) \\
(3.5) \quad &\leq \epsilon_n \|v - u_n\| + \lambda \left[\Phi^{\circ}(u_n; v - u_n) + \mu \Theta^{\circ}(u_n; v - u_n) \right] + \Psi^{\circ}(u_n; u - u_n).
\end{aligned}$$

But, $\Psi \in C^1(X; \mathbb{R})$ and so

$$\Psi^{\circ}(u_n; u - u_n) = \langle \Psi'(u_n), u - u_n \rangle.$$

Thus,

$$(3.6) \quad \epsilon_n \|v - u_n\| + \lambda \left[\Phi^{\circ}(u_n; v - u_n) + \mu \Theta^{\circ}(u_n; v - u_n) \right] + \langle \Psi'(u_n), u - u_n \rangle \geq 0.$$

Since $J_{\lambda, \mu}$ is coercive on X (see Lemma 3.1), we get $\{u_n\}$ is bounded in X and so by passing to a subsequence if necessary, by the Sobolev embedding theorem, we may assume that

$$(3.7) \quad \begin{cases} u_n \rightharpoonup u, & \text{weakly in } X, \\ u_n \rightarrow u, & \text{a.e. in } C([0, T]). \end{cases}$$

Now, we choose $R_0 > 0$ such that for every $n \in \mathbb{N}$

$$\|u_n - u\| < R_0,$$

and sequences $\{v_n^*\}, \{w_n^*\}$ in X^* such that, for every $n \in \mathbb{N}$, $v_n^* \in \partial\Phi(u_n)$, $\{w_n^*\} \in \partial\Theta(u_n)$ and

$$\Phi^{\circ}(u_n; u - u_n) = \langle v_n^*, u - u_n \rangle, \quad \Theta^{\circ}(u_n; u - u_n) = \langle w_n^*, u - u_n \rangle$$

(see Lemma 2.4); by compactness of $\partial\Phi$ and $\partial\Theta$, up to a subsequence, $v_n^* \rightarrow v^* \in X^*$ and $w_n^* \rightarrow w^* \in X^*$.

Fix $\epsilon > 0$, from what was stated above, for $n \in \mathbb{N}$ big enough, one can get

$$\begin{aligned}
\|v_n^* - v^*\|_{X^*} &< \frac{\epsilon}{4|\lambda|R_0}, \quad \|w_n^* - w^*\|_{X^*} < \frac{\epsilon}{4|\lambda||\mu|R_0}, \\
\epsilon_n &< \frac{\epsilon}{4R_0}, \quad \langle \lambda(v^* + \mu w^*), u - u_n \rangle < \frac{\epsilon}{4};
\end{aligned}$$

so, from (3.6) we easily get for $n \in \mathbb{N}$ big enough

$$\langle \Psi'(u_n), u_n - u \rangle < \epsilon,$$

that is

$$\limsup_{n \rightarrow +\infty} \langle \Psi'(u_n), u_n - u \rangle \leq 0.$$

Since Ψ' is of type $(S)_+$ (The proof is similar to [31], Lemma 4.2 and is omitted.), therefore we obtain $u_n \rightarrow u$ in X . Thus, the functional $J_{\lambda, \mu}$ satisfies the (PS)-condition for every $\lambda, \mu \in \mathbb{R}$.

□

Our first result is as follows.

Theorem 3.1. *Assume that A satisfy the hypotheses (F1)-(F2), $j(x, u)$ and $k(x, u)$ satisfy conditions (H1)-(H3) and I_{ij} satisfies conditions (I1), and suppose $j(x, u)$, I_{ij} satisfy the following conditions:*

(H4) *There exists $\alpha > 2$ such that $\limsup_{|u| \rightarrow 0} \frac{\max\{|u^*|: u^* \in \partial j(x, u)\}}{|u|^{\alpha-1}} < \infty$ uniformly for all $x \in [0, T]$;*

(H5) *There exist $0 < \mu < r_0$ where r_0 is a positive constant, $c_0 > 0$ and $M > 0$ such that $c_0 < j(x, u) \leq -\mu j^\circ(x, u; -u)$ for all $u \in \mathbb{R}^N$ with $|u| \geq M$ and $x \in [0, T]$;*

(I2) *I_{ij} ($i = 1, \dots, N; j = 1, \dots, m$) are odd and nonincreasing.*

Then, the problem (1.1) has at least three solutions on X .

proof. Since $\Phi(0) = 0$, we claim that $\Phi(tu) \rightarrow -\infty$ as $t \rightarrow +\infty$. To this end, Let \mathcal{N} be the Lebesgue-null set outside which the hypotheses (H3) and (H5) hold and let $x \in [0, T] \setminus \mathcal{N}$, $u \in \mathbb{R}^N$ with $|u| \geq M$. We set $\mathcal{J}(x, \lambda_1) = j(x, \lambda_1 u)$, $\lambda_1 \in \mathbb{R}$. Clearly, $\mathcal{J}(x, \cdot)$ is locally Lipschitz. By Rademacher's theorem, we see that for every $x \in [0, T]$, $\lambda_1 \rightarrow \mathcal{J}(x, \lambda_1)$ is differentiable a.e. on \mathbb{R} and at a point of differentiability $\lambda_1 \in \mathbb{R}$, we have $\frac{d}{d\lambda_1} \mathcal{J}(x, \lambda_1) \in \partial \mathcal{J}(x, \lambda_1)$. Moreover, by Chain rule, we have $\partial \mathcal{J}(x, \lambda_1) \subset (\partial_u j(x, \lambda_1 u), u)_{\mathbb{R}^N}$, hence $\lambda \partial \mathcal{J}(x, \lambda_1) \subset (\partial_u j(x, \lambda_1 u), \lambda_1 u)_{\mathbb{R}^N}$. From (H5), one can get

$$\lambda_1 \frac{d}{d\lambda_1} \mathcal{J}(x, \lambda_1) \geq \frac{1}{\mu} \mathcal{J}(x, \lambda_1) \implies \frac{\frac{d}{d\lambda_1} \mathcal{J}(x, \lambda_1)}{\mathcal{J}(x, \lambda_1)} \geq \frac{1}{\lambda_1 \mu}.$$

By Integrating from 1 to λ_0 from above inequality, we get $\ln \frac{\mathcal{J}(x, \lambda_0)}{\mathcal{J}(x, 1)} \geq \ln \lambda_0^{\frac{1}{\mu}}$. So, we have proved that for $x \in [0, T] \setminus \mathcal{N}$, $|u| \geq M$ and $\lambda_1 \geq 1$, we have $\lambda_0^{-\frac{1}{\mu}} j(x, \lambda_1 u) \geq \lambda_0^{\frac{1}{\mu}} j(x, u)$.

Let $z(x) = \min\{j(x, u) : |u| = M\}$, clearly $z \in L^2([0, T], \mathbb{R}^+)$ and $z(x) \geq c_0$ for every $x \in [0, T]$. Therefore, for every $x \in [0, T] \setminus \mathcal{N}$ and $|u| \geq M$, we have

$$(3.8) \quad j(x, u) = j(x, |u| M^{-1} M u |u|^{-1}) \geq \left(\frac{|u|}{M}\right)^{\frac{1}{\mu}} j\left(x, \frac{u}{|u|} M\right) \geq z(x) \left(\frac{|u|}{M}\right)^{\frac{1}{\mu}}.$$

On the other hand, by means of the equivalence between two norms in finite-dimensional space, for any finite-dimensional subspace $U \subset X$ and any $u \in U$, there exists a constant $C > 0$ such that

$$\|u\|_\delta = \left(\int_0^T |u(x)|^\delta dx\right)^{\frac{1}{\delta}} \geq C \|u\|, \quad \delta \geq 1.$$

Then, by (3.2) and (3.8), there exists a positive constant C_1 such that

$$\begin{aligned} \Phi(u) &= -\int_0^T j(x, u(x)) dx \leq -\int_0^T z(x) \left(\frac{|u(x)|}{M}\right)^{\frac{1}{\mu}} dx \\ &\leq -c_0 \left(\frac{1}{M}\right)^{\frac{1}{\mu}} \|u\|_{\frac{1}{\mu}}^{\frac{1}{\mu}} \leq -c_0 C \left(\frac{1}{M}\right)^{\frac{1}{\mu}} \|u\|_{\frac{1}{\mu}}^{\frac{1}{\mu}} \end{aligned}$$

Since $0 < \mu < r_0$, then for any $u \in X \setminus \{0\}$, we have $\Phi(tu) \rightarrow -\infty$ as $t \rightarrow +\infty$. Hence the claim is true. Then, for large $t_0 > 0$, we take $u_0 = t_0 u$ with $u \in X \setminus \{0\}$ fixed, then $\Phi(u_0) < 0$, that is, $u_0 \in \Phi^{-1}(-\infty, 0)$, hence that $\mathbb{R}_0^- \subset (\inf \Phi, \sup \Phi)$ follows from the locally Lipschitz continuity of Φ .

If we denote

$$(3.9) \quad \lambda^* = \varphi_1(0) = \inf_{u \in \Phi^{-1}(I_0)} \frac{-\Psi(u)}{\Phi(u)}, \quad I_0 = (-\infty, 0).$$

By the above argument, we see that λ^* is well defined.

Similar to the proof of (4.5) in [32], one can get

$$(3.10) \quad \limsup_{r \rightarrow 0^-} \varphi_1(r) \leq \varphi_1(0) = \lambda^*.$$

On the other hand, since I_{ij} ($i = 1, \dots, N; j = 1, \dots, m$) are odd and nonincreasing, one has that the $\int_0^y I_{ij}(s)ds$ are even and $\int_0^y I_{ij}(s)ds \leq 0$ for any $y \geq 0$; thus

$$(3.11) \quad \sum_{i=1}^N \int_0^{u^i(x_j)} I_{ij}(s)ds \leq 0.$$

Also, from (H3) and (H4), we can deduce that $|j(x, u)| \leq C_1|u|^\alpha$, for every $u \in \mathbb{R}$, where $C_1 > 0$ is a constant. So, for every $u \in X$, it is easy to deduce that $|\Phi(u)| \leq C_2\|u\|^\alpha$, where $C_2 > 0$ is a constant. Therefore, given $r < 0$ and $u \in \Phi^{-1}(r)$, by (3.11), we have

$$(3.12) \quad -r = -\Phi(u) \leq C_2\|u\|^\alpha = C_3 \left(\frac{\|u\|^2}{2} \right)^{\frac{\alpha}{2}} \leq C_3(\Psi(u))^{\frac{\alpha}{2}},$$

where $C_3 = 2^{\frac{\alpha}{2}}C_2$. Since $0 \in \Phi^{-1}((r, +\infty))$, by definition on $\varphi_2(r)$ and (3.12), we have

$$\varphi_2(r) \geq \frac{1}{|r|} \inf_{v \in \Phi^{-1}(r)} \Psi(v) \geq C_3^{-\frac{2}{\alpha}} |r|^{\frac{2}{\alpha}-1}.$$

In view of $\alpha > 2$, so that the above inequalities imply that $\lim_{r \rightarrow 0^-} \varphi_2(r) = +\infty$. Consequently, we have proved that

$$\lim_{r \rightarrow 0^-} \varphi_1(r) = \varphi_1(0) = \lambda^* < \lim_{r \rightarrow 0^-} \varphi_2(r) = +\infty.$$

This yields that for all integers $n \geq n^* = 2 + [\lambda^*]$ there exists a number $r_n < 0$ so close to zero such that $\varphi_1(r_n) < \lambda^* + \frac{1}{n} < n < \varphi_2(r_n)$. Hence, since $\Lambda = \mathbb{R}$, by Theorem 2.1, for every compact interval

$$[a, b] \in (\lambda^*, \infty) = \bigcup_{n=n^*}^{\infty} \left[\lambda^* + \frac{1}{n}, n \right] \subset \bigcup_{n=n^*}^{\infty} (\varphi_1(r_n), \varphi_2(r_n)) \cap \Lambda,$$

there exists $\delta > 0$ such that problem (1.1) admits at least three solutions for every $\lambda \in [a, b]$ and $\mu \in (-\delta, \delta)$. Therefore, we finish the proof. \square

Now, we introduce some notations.

$$K_1 = \frac{T\left(\frac{\xi_1}{\eta_1} + \frac{\xi_2}{\eta_2}\right) + \frac{\alpha_1}{\alpha_2}\xi_1 + \frac{\beta_1}{\beta_2}\xi_2}{T\left(\frac{1}{\eta_1} + \frac{1}{\eta_2} - 1\right)},$$

$$K_2 = |\xi_1|^p \int_0^{\frac{T}{\eta_1}} \rho(x) dx + |K_1|^p \int_{\frac{T}{\eta_1}}^{T-\frac{T}{\eta_2}} \rho(x) dx + |\xi_2|^p \int_{T-\frac{T}{\eta_2}}^T \rho(x) dx,$$

$$K_3 = \left\{ \left(\frac{T}{\eta_1} + \frac{\alpha_1}{\alpha_2}\right)|\xi_1|, \frac{\alpha_1}{\alpha_2}|\xi_1|, \left(\frac{T}{\eta_2} + \frac{\beta_1}{\beta_2}\right)|\xi_2|, \frac{\beta_1}{\beta_2}|\xi_2| \right\}.$$

In the following result we replace condition (H5) by conditions (H6).

Theorem 3.2. *Assume that there exist four positive constants ξ_1, ξ_2, η_1 and η_2 with $\eta_1\eta_2 < \eta_1\eta_2$ and the hypotheses (F1)-(F2), (H1)-(H4) and (I1)-(I2) hold, suppose $j(x, u)$ satisfies the following condition:*

(H6) *there exists $\sigma \in (0, r_0)$ where r_0 is a positive constant, such that $\lim_{|u| \rightarrow +\infty} \frac{j(x, u)}{|u|^\sigma} = +\infty$ uniformly for all $x \in [0, T]$;*

Then, the problem (1.1) has at least three solutions on X .

proof. From the proof of Theorem 3.1, we only need to prove that $\Phi^{-1}(-\infty, 0) \neq \emptyset$. To this end, we prove that there exists $u_0 \in X$ such that $\Phi(u_0) < 0$. By (H6), for any constant $\varrho_3 > 0$, there exists $\delta_3 > 0$ such that

$$j(x, u) \geq \varrho_3|u|^\sigma, \quad \text{for all } |u| \geq \delta_3, x \in [0, T].$$

It follows from (H2), (H3) and the Lebourg's mean value theorem that

$$(3.13) \quad j(x, u) \geq \varrho_3|u|^\sigma - \varrho_3\delta_3^2 - \bar{a}_4(x), \quad \text{for all } u \in \mathbb{R}^N, x \in [0, T],$$

where $\bar{a}_4(x) \in L^1([0, T], \mathbb{R}^+)$. Set $\mathcal{E} = (1, 0, \dots, 0) \in \mathbb{R}^N$, therefore, by (2.6), (3.13) and (H3), choose $u_0(x)$ as follow

$$u_0(x) = \begin{cases} \xi_1\left(x + \frac{\alpha_1}{\alpha_2}\right)\mathcal{E}, & 0 \leq x < \frac{T}{\eta_1}, \\ \xi_1\left(\frac{T}{\eta_1} + \frac{\alpha_1}{\alpha_2}\right)\mathcal{E} + K_1\left(x - \frac{T}{\eta_1}\right)\mathcal{E}, & \frac{T}{\eta_1} \leq x \leq T - \frac{T}{\eta_2} \\ \xi_2\left(x - \frac{\beta_1}{\beta_2} - T\right)\mathcal{E}, & T - \frac{T}{\eta_2} < x \leq T. \end{cases}$$

Clearly, $u_0 \in H_T^1$. In view of

$$\int_0^T \rho(x)|\dot{u}_0(x)|^2 dx = K_2, \quad 0 \leq \int_0^T |u_0(x)|^\sigma dx \leq TK_3^\sigma.$$

Thus, by Holder's inequality, we have

$$\begin{aligned} \Phi(su_0) &= - \int_0^T j(x, su_0(x)) dx \leq -s^\sigma \varrho_3 \int_0^T |u_0(x)|^\sigma dx + C_4 \\ &\leq -\varrho_3 TK_3^\sigma s^\sigma + C_4 \end{aligned}$$

Here C_4 is a positive constant. Then for large $s_0 > 0$, we take $u_1 = s_0 u_0$, then $\Phi(u_1) < 0$. Therefore, we complete the proof. \square

Theorem 3.3. *Assume that the hypotheses (F1)-(F2), (H1)-(H4) and (I1)-(I2) hold, suppose $j(x, u)$ satisfies the following condition:*

(H7) *There exists $1 < \beta < 2$ such that $\liminf_{|u| \rightarrow \infty} \frac{\max\{|u^*|: u^* \in \partial j(x, u)\}}{|u|^{\beta-1}} > 0$ uniformly for all $x \in [0, T]$.*

Then, the problem (1.1) has at least three solutions on X .

proof. From the proof of Theorem 3.1, we only need to prove that $\Phi^{-1}(-\infty, 0) \neq \emptyset$. To our purpose, from (H3) and (H7) we have $j(x, u) \geq C_5|u|^\beta - C_6$, where C_5 and C_6 are positive constants. Thus, one can get

$$\Phi(u) = - \int_0^T j(x, u(x)) dx \leq -C_5 \int_0^T |u(x)|^\beta dx + C_6 T = -C_5 \|u\|_{L^\beta(0, T; \mathbb{R}^N)}^\beta dx + C_6 T.$$

So,

$$\lim_{u \in X, \|u\|_{L^\beta(0, T; \mathbb{R}^N)} \rightarrow \infty} \Phi(u) = -\infty,$$

So that $\mathbb{R}_0^- \subset (\inf \Phi, \sup \Phi)$ follows from the locally Lipschitz continuity of Φ . \square

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