



OPTIMAL INEQUALITIES FOR SUBMANIFOLDS IN AN (ε) -ALMOST PARA-CONTACT MANIFOLDS

MOHAMMED DANISH SIDDIQI, GHODRATALLAH FASIHI-RAMANDI*, AND MOHAMMED HASAN

ABSTRACT. The present research paper studies the optimal inequalities for the Casorati curvature of submanifolds in an (ε) -almost para-contact manifold, precisely (ε) -Kenmotsu manifolds endowed with a semi-symmetric metric connection (briefly *SSM*) by adopting the T. Oprea's optimization technique.

MSC(2010): 53C15 ; 53C25.

Keywords: (ε) -Kenmotsu manifold, semi-symmetric metric connection, Casorati curvatures, submanifolds.

1. Introduction and Background

In 1971, Kenmotsu investigated a class of contact Riemannian manifolds, named Kenmostu manifolds, which satisfy some special conditions [?]. After that, Kenmotsu manifolds have been discussed by Pathak and De [?], Jun et al. [?] and many authors. Moreover, Bejancu and Duggal [?] introduced the notion of (ε) -Sasakian manifolds, which later on, Xufeng and Xiaoli [?] established that these manifolds are real hyper-surface of Kaehlerian manifolds. An (ε) -almost para-contact manifold is introduced by Tripathi et al. [?]. While the concept of (ε) -Kenmotsu manifolds were developed by De and Sarkar [?]. They showed that these structures exist with indefinite metrics.

In 1924, Friedmann and Schouten [?] introduced the concept of semi-symmetric linear connections on a differentiable manifold in 1924. Therefore, Bartolotti [?] gave a geometrical meaning of such connections. In 1932, Hayden [?] defined the notion of semi-symmetric metric connections. In [?], Yano initiated a systematic study of the semi-symmetric metric connections in a Riemannian manifold and other structures were further studied by various authors such as Sharfuddin Ahmad and Hussain [?], Tripathi [?], Hirićă, Nicolescu ([?], [?]) and Siddiqi et al. ([?],[?], [?]).

Let ∇ be a linear connection in an n -dimensional differentiable manifold M . The torsion tensor T of ∇ is given by

$$T(E, F) = \nabla_E F - \nabla_F E - [E, F].$$

The connection ∇ is said to be a symmetric if its torsion tensor T vanishes, otherwise it is said non-symmetric. Let (M, g) be a Riemannian manifold, the connection ∇ is said to be

Date: Received: January 21, 2021, Accepted: May 18, 2021.

*Corresponding author.

a metric connection if $\nabla g = 0$, otherwise it is said non-metric. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection.

A linear connection ∇ is said to be semi-symmetric connection if its torsion tensor T is of the form

$$T(E, F) = \eta(F)E - \eta(E)F,$$

where η is a 1-form.

On other side, the Casorati curvature C of a submanifold M with dimension n of a Riemannian manifold \bar{M} is an extrinsic invariant described as the normalized square of the length of the second fundamental form h of the submanifold M . In [?], Lee et al. introduced the normalized δ' -Casorati curvatures $\delta'_C(n-1)$ and $\bar{\delta}'_C(n-1)$ by

$$[\delta'_C(n-1)]_x = \frac{1}{2}C_x + \frac{n+1}{2n} \inf\{C(L)|L \text{ a hyperplane of } T_xM\}, \text{ and}$$

$$[\bar{\delta}'_C(n-1)]_x = 2C_x - \frac{2n-1}{2n} \sup\{C(L)|L \text{ a hyperplane of } T_xM\},$$

where $x \in M$, and proved some inequalities involving these δ' -invariants for submanifolds in real space forms endowed with a semi-symmetric metric connection. Since then many researchers obtained such inequalities for different submanifolds in different ambient spaces (for example [?], [?], [?]).

By using T. Oprea's optimization method on Riemannian submanifolds, we will establish some inequalities in terms of $\delta'_C(n-1)$ for submanifolds of (ε) -Kenmotsu manifolds with respect to a semi-symmetric metric connection (in short *SSM*)[?].

2. Preliminaries

An odd dimensional ($m = 2n + 1$) smooth manifold (\bar{M}^m, g) is said to be an (ε) -almost contact metric manifold [?], if it admits a $(1, 1)$ -tensor field ϕ , a structure vector field ξ , a 1-form η and an indefinite metric g such that

$$(2.1) \quad \phi^2 E = -E + \eta(E)\xi,$$

$$(2.2) \quad \eta(\xi) = 1,$$

$$(2.3) \quad g(\xi, \xi) = \varepsilon,$$

$$(2.4) \quad \eta(E) = \varepsilon g(E, \xi),$$

$$(2.5) \quad g(\phi E, \phi F) = g(E, F) - \varepsilon \eta(E)\eta(F)$$

for all vector fields E, F on $\chi(\bar{M})$, where ε is 1 or -1 depending on ξ is space like or time like vector field and rank ϕ is $(n-1)$. If

$$(2.6) \quad d\eta(E, F) = g(E, \phi F),$$

for every $E, F \in \chi(\bar{M})$, where $\chi(\bar{M})$ is space of smooth vector fields on \bar{M} , then we say that $\bar{M}(\phi, \xi, \eta, g, \varepsilon)$ is an (ε) -almost contact metric manifold. Also, we have

$$(2.7) \quad \phi\xi = 0, \quad \eta(\phi E) = 0.$$

If an (ε) -contact metric manifold satisfies

$$(2.8) \quad (\nabla_E \phi)(F) = -g(E, \phi F) - \varepsilon \eta(F)\phi E,$$

where ∇ denotes the Levi-Civita connection with respect to g , then \bar{M} is called an (ε) -Kenmotsu manifold [?].

An (ε) -almost contact metric manifold is an (ε) -Kenmotsu if and only if

$$(2.9) \quad \nabla_E \xi = \varepsilon[E - \eta(E)\xi].$$

Moreover, the curvature tensor R , the Ricci tensor S of the (ε) -Kenmotsu manifold \bar{M} satisfy [?]

$$(2.10) \quad (\nabla_E \eta)F = [g(E, F) - \varepsilon\eta(E)\eta(F)], \quad (\nabla_\xi \eta)F = 0,$$

$$(2.11) \quad R(E, F)\xi = \eta(E)F - \eta(F)E,$$

$$(2.12) \quad R(\xi, E)F = \eta(F)E - \varepsilon g(E, F)\xi,$$

$$(2.13) \quad R(\xi, E)\xi = -R(E, \xi)\xi = E - \eta(E)\xi,$$

$$(2.14) \quad \eta(R(E, F)G) = \varepsilon[g(E, G)\eta(F) - g(F, G)\eta(E)],$$

$$(2.15) \quad S(E, \xi) = -(m - 1)\eta(E),$$

Remark 2.1. Note that if $\varepsilon = 1$ then an (ε) -Kenmotsu manifold is a usual Kenmotsu manifold.

Let \bar{M} be an (ε) -Kenmotsu manifold and ∇ be the Levi-Civita connection on \bar{M} . The semi-symmetric metric connection $\bar{\nabla}$ on \bar{M} is given by

$$(2.16) \quad \bar{\nabla}_E F = \nabla_E F + \eta(F)E - g(E, F)\xi$$

where $E, F, G \in \chi(M)$.

3. Curvature tensor on an (ε) -Kenmotsu manifold with semi-symmetric metric connection

Let the curvature tensor \bar{R} of an (ε) -Kenmotsu manifold \bar{M} with respect to the semi-symmetric metric connection $\bar{\nabla}$ is defined by

$$(3.1) \quad \bar{R}(E, F)G = \bar{\nabla}_E \bar{\nabla}_F G - \bar{\nabla}_F \bar{\nabla}_E G - \bar{\nabla}_{[E, F]} G.$$

By virtue of (??) and (??), we have

$$(3.2) \quad \begin{aligned} \bar{R}(E, F)G &= (\nabla_E \nabla_F G - \nabla_F \nabla_E G - \nabla_{[E, F]} G) + [(\nabla_E \eta)(G)F - (\nabla_F \eta)(G)E] \\ &\quad + [g(E, G)\nabla_F \xi - g(F, G)\nabla_E \xi] + \eta(G)[\eta(F)E - \eta(E)F] \\ &\quad + [g(E, G)F - g(F, G)E] + [g(F, G)\eta(E) - g(E, G)\eta(F)]\xi. \end{aligned}$$

In view of (2.4), (2.9) and (2.10), we get

$$(3.3) \quad \begin{aligned} \bar{R}(E, F)G &= R(E, F)G + (2 + \varepsilon)[g(E, G)F - g(F, G)E] \\ &\quad + (1 + \varepsilon)[g(F, G)\eta(E) - g(E, G)\eta(F)]\xi \\ &\quad + (1 + \varepsilon)[\eta(F)E - \eta(E)F]\eta(G), \end{aligned}$$

where R is the Riemannian curvature tensor of Riemannian connection ∇ .

Now, contracting E in (3.3), we get

$$(3.4) \quad \bar{S}(F, G) = S(F, G) + [(\varepsilon + 2)(\varepsilon - m) + 2]g(F, G) + (1 + \varepsilon)(m - 2\varepsilon)\eta(F)\eta(G),$$

where \bar{S} and S are the Ricci tensors of the connections $\bar{\nabla}$ and ∇ , respectively on \bar{M} .

Contracting again F and G in (3.10), it follows that

$$(3.5) \quad \bar{\tau} = \tau + m[(\varepsilon + 2)(\varepsilon - m) + 2] + (1 + \varepsilon)(m - 2\varepsilon),$$

where $\bar{\tau}$ and τ are the scalar curvatures of the connections $\bar{\nabla}$ and ∇ , respectively on M .

4. CASORATI CURVATURES

Let M be an n -dimensional submanifold of an m -dimensional (ε) -Kenmotsu manifold \bar{M} with the SSM connection $\bar{\nabla}$ of induced metric g . We represent the induced connections on the tangent bundle TM and TM^\perp of M by ∇^M and ∇^{M^\perp} , respectively and denote by h the second fundamental form of M . For any $E, F \in T_x M$, and $N \in TM^\perp$, $x \in M$, we recall the Gauss and Weingarten formulas by

$$\nabla_E F = \nabla_E^M F + h(E, F),$$

and

$$\nabla_E^\perp N = -A_N E + \nabla_E^{M^\perp} N,$$

where A_N is used for notation of the shape operator of M with respect to N . The following equation is well-known:

$$g(A_N E, F) = g(h(E, F), N).$$

We also recall the equation of Gauss by

$$\begin{aligned} R(E, F, G, H) &= R^M(E, F, G, H) - g(h(E, H), h(F, G)) \\ &\quad + g(h(E, G), h(F, H)), \end{aligned}$$

for any $E, F, G, H \in T_x M$, $x \in M$. Here R^M is the Riemannian curvature tensor with respect to ∇^M .

For a surface in \mathbb{E}^3 the Casorati curvature is defined as the normalized sum of the squared principal curvatures. This curvature was preferred by Casorati over the traditional Gauss curvature because the Casorati curvature vanishes if and only if both principal curvatures are zero at the same time and thus corresponds better with the common intuition of curvature.

We choose an orthonormal tangent frame $\{e_1, \dots, e_n\}$ and an orthonormal normal frame $\{e_{n+1}, \dots, e_m\}$ of M in an (ε) -Kenmotsu manifold \bar{M} . Then the scalar curvature τ at $x \in M$ is defined by

$$\tau(x) = \sum_{i < j} K(e_i \wedge e_j),$$

where $K(e_i \wedge e_j)$, $1 \leq i < j \leq n$, denotes the sectional curvature of M associated with a plane section spanned by e_i and e_j .

The normalized scalar curvature ρ of M is defined as

$$\rho = \frac{2\tau}{n(n-1)}.$$

Let L be a subspace of $T_x M$, $x \in M$ of dimension $p \geq 2$ and $\{e_1, \dots, e_n\}$ an orthonormal basis of L . The scalar curvature $\tau(L)$ of the p -plane section L is given by

$$\tau(L) = \sum_{1 \leq r < s \leq p} K(e_r \wedge e_s).$$

The squared norm of h over the dimension n is called the Casorati curvature C of the submanifold M in \bar{M} , that is,

$$nC = \sum_{a=n+1}^m \left(\sum_{i,j=1}^n (h_{ij}^a)^2 \right),$$

whereby $h_{ij}^a = g(h(e_i, e_j), e_a)$ are the components of the second fundamental form with respect to given orthonormal bases. Also, the Casorati curvature $C(L)$ of the subspace L is defined as

$$pC = \sum_{a=n+1}^m \left(\sum_{i,j=1}^p (h_{ij}^a)^2 \right).$$

The squared mean curvature of the submanifold M in \bar{M} is given by

$$n^2 \|\mathcal{H}\|^2 = \sum_{a=n+1}^m \left(\sum_{i=1}^n h_{ii}^a \right)^2.$$

From the Gauss equation, the following relation between the scalar curvature, the squared mean curvature and the Casorati curvature holds:

$$(4.1) \quad \begin{aligned} 2\tau &= n(n-1) + n[(\varepsilon+2)(\varepsilon-n) + 2] + (1+\varepsilon)(n-2\varepsilon) \\ &+ n^2 \|\mathcal{H}\|^2 - nC. \end{aligned}$$

5. UPPER BOUNDS FOR δ' -CASORATI CURVATURE

Let (\bar{B}, \bar{g}) be an (ε) -almost contact metric manifold, B be an (ε) -almost contact metric manifold of it, g be the metric induced on B by \bar{g} and $f : (\bar{B}, \bar{g}) \rightarrow (\mathbb{R}, g_0)$ be a differentiable function, where g_0 stands for the standard metric on \mathbb{R} .

Following [?, ?, ?], we consider the constrained extremum problem $\min_{x \in B} f(x)$, then we have the following:

Lemma 5.1. *If $x_0 \in B$ is the solution of the above problem, then*

- (1) $(gradf)(x_0) \in T_{x_0}^\perp B$,
- (2) the bilinear form

$$\mathcal{A} : T_{x_0} B \times T_{x_0} B \rightarrow \mathbb{R},$$

$$\mathcal{A}(X, Y) = Hess_f(X, Y) + \bar{g}(h(X, Y), (gradf)(x_0))$$

is positive semi-definite, where h is the second fundamental form of B in \bar{B} .

6. MAIN RESULT

Theorem 6.1. *Let M be an n -dimensional, $n > 2$, submanifold of an m -dimensional (ε) -Kenmotsu manifold \bar{M} with a SSM connection. Then the normalized δ' -Casorati curvature $\delta'_C(n-1)$ satisfies*

$$\rho \leq \delta'_C(n-1) + 1 + \frac{1}{(n-1)}[(\varepsilon+2)(\varepsilon-n)+2] + (1+\varepsilon)(n-2\varepsilon).$$

Moreover, the equality case holds if and only if M^n is an invariantly quasi-umbilical submanifold with trivial normal connection in M^m , such that with respect to suitable orthonormal tangent frame $\{e_1, \dots, e_n\}$ and normal orthonormal frame $\{e_{n+1}, \dots, e_m\}$, the shape operators $A_b = Ae_b$, $b \in \{n+1, \dots, m\}$, take the following forms:

$$(6.1) \quad A_{n+1} = \begin{pmatrix} t_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & t_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & t_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & t_1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 2t_1 \end{pmatrix}, \quad A_{n+2} = \dots = A_m = 0.$$

Proof. In light of (??), let us define the following function \mathcal{F} as a quadratic polynomial in terms of the components of the second fundamental form:

$$\mathcal{F} = \frac{1}{2}(n-1)[nC + (n+1)C(L)] - 2\tau + n(n-1) + n[(\varepsilon+2)(\varepsilon-n)+2] + (1+\varepsilon)(n-2\varepsilon).$$

By saying that L is spanned by $\{e_1, \dots, e_{n-1}\}$ (without loss of generality), one can easily deduce

$$\begin{aligned} \mathcal{F} &= \frac{1}{2}(n-1) \sum_{a=n+1}^m \left[\sum_{i,j=1}^n (h_{ij}^a)^2 \right] + \frac{1}{2}(n+1) \sum_{a=n+1}^m \left[\sum_{i,j=1}^{n-1} (h_{ij}^a)^2 \right] - 2\tau \\ &\quad + n(n-1) + n[(\varepsilon+2)(\varepsilon-n)+2] + (1+\varepsilon)(n-2\varepsilon). \end{aligned}$$

By substituting equation (??) in the last equation, we have

$$(6.2) \quad \begin{aligned} \mathcal{F} &= \frac{1}{2}(n-1) \sum_{a=n+1}^m \left[\sum_{i,j=1}^n (h_{ij}^a)^2 \right] + \frac{1}{2}(n+1) \sum_{a=n+1}^m \left[\sum_{i,j=1}^{n-1} (h_{ij}^a)^2 \right] \\ &\quad + n^2 \|\mathcal{H}\|^2 - nC. \end{aligned}$$

The simple modification in (??) gives us the following:

$$(6.3) \quad \begin{aligned} \mathcal{F} &= \frac{1}{2}(n+1) \left\{ \sum_{a=n+1}^m \left[\sum_{i,j=1}^n (h_{ij}^a)^2 \right] + \sum_{a=n+1}^m \left[\sum_{i,j=1}^{n-1} (h_{ij}^a)^2 \right] \right\} \\ &\quad - \sum_{a=n+1}^m \left(\sum_{i=1}^n h_{ii}^a \right)^2. \end{aligned}$$

We rewrite the equation (??) as follows:

$$\begin{aligned} \mathcal{F} &= \sum_{a=n+1}^m \sum_{i=1}^{n-1} \left[n(h_{ii}^a)^2 + (n+1)(h_{nn}^a)^2 \right] \\ &\quad + \sum_{a=n+1}^m \left[2(n+1) \sum_{1 \leq i < j \leq n-1} (h_{ij}^a)^2 \right. \\ &\quad \left. - 2 \sum_{1 \leq i < j \leq n} h_{ii}^a h_{jj}^a + \frac{1}{2}(n-1)(h_{nn}^a)^2 \right]. \end{aligned}$$

By eliminating second and third terms in the last equation, one gets the following inequality:

$$(6.4) \quad \mathcal{F} \geq \sum_{a=n+1}^m \left[\sum_{i=1}^{n-1} n(h_{ii}^a)^2 - 2 \sum_{1 \leq i < j \leq n} h_{ii}^a h_{jj}^a + \frac{1}{2}(n-1)(h_{nn}^a)^2 \right].$$

For $a = n+1, \dots, m$, we define the quadratic form

$$\lambda^a : \mathbb{R}^n \rightarrow \mathbb{R}$$

by

$$\lambda^a(h_{11}^a, \dots, h_{nn}^a) = \sum_{i=1}^{n-1} n(h_{ii}^a)^2 - 2 \sum_{1 \leq i < j \leq n} h_{ii}^a h_{jj}^a + \frac{1}{2}(n-1)(h_{nn}^a)^2,$$

and the constrained extremum problem $\min \lambda^a$ subject to $B : \sum_{i=1}^n h_{ii}^a = c^a$, where c^a is a real constant.

The partial derivatives of λ^a with respect to $h_{11}^a, \dots, h_{nn}^a$ are given below:

$$(6.5) \quad \frac{\partial \lambda^a}{\partial h_{11}^a} = 2nh_{11}^a - 2 \sum_{i=1}^n h_{ii}^a,$$

$$(6.6) \quad \frac{\partial \lambda^a}{\partial h_{22}^a} = 2nh_{22}^a - 2h_{11}^a - 2 \sum_{i=3}^n h_{ii}^a,$$

⋮

$$(6.7) \quad \frac{\partial \lambda^a}{\partial h_{n-1, n-1}^a} = 2nh_{n-1, n-1}^a - 2h_{nn}^a - 2 \sum_{i=1}^{n-2} h_{ii}^a,$$

$$(6.8) \quad \frac{\partial \lambda^a}{\partial h_{nn}^a} = (n-1)h_{nn}^a - 2 \sum_{i=1}^{n-1} h_{ii}^a.$$

For an optimal solution $(h_{11}^a, h_{22}^a, \dots, h_{nn}^a)$ of the problem in question, the vector $grad \lambda^a$ is normal at B , that is, it is collinear with the vector $(1, 1, \dots, 1)$. From the above partial derivatives, it follows that a critical point of the considered problem has the following form:

$$(6.9) \quad (h_{11}^a, h_{22}^a, \dots, h_{nn}^a) = (f^a, f^a, \dots, f^a, 2f^a).$$

By using (??), we have

$$(6.10) \quad h_{11}^a = h_{22}^a = \dots = h_{n-1, n-1}^a = \frac{1}{n+1}c^a, \quad h_{nn}^a = \frac{2}{n+1}c^a.$$

Now, let us denote the second fundamental form of B in \mathbb{R}^n by h' . We fix an arbitrary point $x_0 \in B$. The bilinear form

$$\mathcal{A} : T_{x_0}B \times T_{x_0}B \longrightarrow \mathbb{R}$$

is defined by

$$(6.11) \quad \mathcal{A}(X, Y) = Hess_{\lambda^a}(X, Y) + \langle h'(X, Y), (grad \lambda^a)(x_0) \rangle .$$

In the standard frame of \mathbb{R}^n , the Hessian of λ^a has the matrix:

$$(6.12) \quad Hess_{\lambda^a} = 2 \begin{pmatrix} n & -1 & -1 & \dots & -1 & -1 \\ -1 & n & -1 & \dots & -1 & -1 \\ -1 & -1 & n & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \dots & n & -1 \\ -1 & -1 & -1 & \dots & -1 & \frac{n-1}{2} \end{pmatrix} .$$

Since, B is totally geodesic in \mathbb{R}^n and we consider a vector $X = (X_1, \dots, X_n)$ tangent to B at x_0 , then by (??), we have

$$\mathcal{A}(X, X) = (n+1) \left(2 \sum_{i=1}^{n-1} X_i^2 + X_n^2 \right) - 2 \left(\sum_{i=1}^n X_i \right)^2 \geq 0.$$

Thus, the critical point given by (??) is a global minimum point, here we used Lemma ??- Inserting (??) in (??), we have $\mathcal{F} \geq 0$. It is easy to derive the following inequality:

$$(6.13) \quad \rho \leq \delta'_C(n-1) + 1 + \frac{1}{(n-1)} [(\varepsilon+2)(\varepsilon-n) + 2] + (1+\varepsilon)(n-2\varepsilon).$$

The equality case of (??) holds if and only if we have the equality in all the previous inequalities and we find

$$\begin{aligned} h_{ij}^a &= 0, \quad i \neq j, \quad \forall a, \\ h_{nn}^a &= 2h_{11}^a = 2h_{22}^a = \dots = 2h_{n-1, n-1}^a, \quad \forall a. \end{aligned}$$

□

Similarly, for the normalized δ' -Casorati curvature $\bar{\delta}'_C(n-1)$ from equation (??), we have.

Theorem 6.2. *Let M be an n -dimensional, $n > 2$, submanifold of an m -dimensional (ε) -Kenmotsu manifold \bar{M} with a SSM connection. Then the normalized δ' -Casorati curvature $\bar{\delta}'_C(n-1)$ satisfies*

$$\rho \leq \bar{\delta}'_C(n-1) + 1 + \frac{1}{(n-1)} [(\varepsilon+2)(\varepsilon-n) + 2] + (1+\varepsilon)(n-2\varepsilon).$$

Moreover, the equality case holds if and only if M^n is an invariantly quasi-umbilical submanifold with trivial normal connection in \bar{M}^m , such that with respect to suitable orthonormal

tangent frame $\{e_1, \dots, e_n\}$ and normal orthonormal frame $\{e_{n+1}, \dots, e_m\}$, the shape operators $A_b = Ae_b$, $b \in \{n + 1, \dots, m\}$, take the following forms:

$$(6.14) \quad A_{n+1} = 2 \begin{pmatrix} t_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & t_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & t_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & t_1 & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{t_1}{2} \end{pmatrix}, \quad A_{n+2} = \dots = A_m = 0.$$

As a consequence of Theorems ?? and ??, we give the following results:

Corollary 6.3. *Let M be an n -dimensional, $n > 2$, submanifold of an m -dimensional of a usual Kenmotsu manifold \bar{M} with a SSM connection. Then*

$$\rho \leq \delta'_C(n - 1) + \frac{(5 - n)}{(n - 1)} + 2n - 3.$$

Corollary 6.4. *Let M be an n -dimensional, $n > 2$, submanifold of an m -dimensional of a usual Kenmotsu manifold \bar{M} with a SSM connection. Then*

$$\rho \leq \bar{\delta}'_C(n - 1) + \frac{(5 - n)}{(n - 1)} + 2n - 3.$$

The equality case holds in Corollaries ?? and ?? if and only if M^n is an invariantly quasi-umbilical submanifold with trivial normal connection in \bar{M}^m , such that with respect to suitable orthonormal tangent frame $\{e_1, \dots, e_n\}$ and normal orthonormal frame $\{e_{n+1}, \dots, e_m\}$, the shape operators $A_b = Ae_b$, $b \in \{n + 1, \dots, m\}$, take forms as in (??) and (??).

Example 6.5. We consider the three dimensional manifold $M = [(x, y, z) \in \mathbb{R}^3 \mid z \neq 0]$, where (x, y, z) are the cartesian coordinates in \mathbb{R}^3 . Choose the vector fields

$$e_1 = z \frac{\partial}{\partial x}, \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = -z \frac{\partial}{\partial z},$$

which are linearly independent at each point of M . Let g be the Riemannian metric define by

$$g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = \varepsilon,$$

where $\varepsilon = \pm 1$.

Let η be the 1-form defined by $\eta(Z) = \varepsilon g(Z, e_3)$ for any vector field Z on M . Let ϕ be the $(1, 1)$ -tensor field defined by $\phi(e_1) = -e_2$, $\phi(e_2) = e_1$, $\phi(e_3) = 0$. Then by the linearity property of ϕ and g , we have

$$\phi^2 Z = -Z + \eta(Z)e_3, \quad \eta(e_3) = 1 \text{ and } g(\phi Z, \phi W) = g(Z, W) - \varepsilon \eta(Z)\eta(W)$$

for any vector fields Z, W on M .

Let ∇ be the Levi-Civita connection with respect to the metric g . Then we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = \varepsilon e_1, \quad [e_2, e_3] = \varepsilon e_2.$$

The Riemannian connection ∇ with respect to the metric g is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) - g([Y, Z], X) \\ + g([Z, X], Y).$$

From above equation which is known as Koszul's formula, we have

$$\nabla_{e_1} e_3 = \varepsilon e_1, \quad \nabla_{e_2} e_3 = \varepsilon e_2, \quad \nabla_{e_3} e_3 = 0,$$

$$\nabla_{e_1} e_2 = 0, \quad \nabla_{e_2} e_2 = -\varepsilon e_3, \quad \nabla_{e_3} e_2 = 0,$$

$$\nabla_{e_1} e_1 = -\varepsilon e_3, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_3} e_1 = 0.$$

Using the above relations, for any vector field X on M , we have

$$\nabla_X \xi = \varepsilon[X - \eta(X)\xi]$$

for $\xi = e_3$. Hence the manifold M under consideration is an (ε) -Kenmotsu manifold of dimension three.

Let $\bar{\nabla}$ be a semi-symmetric metric connection. From (??) we obtain:

$$(6.15) \quad \bar{\nabla}_{e_1} e_3 = (1 + \varepsilon)e_1, \quad \bar{\nabla}_{e_2} e_3 = (1 + \varepsilon)e_2, \quad \bar{\nabla}_{e_3} e_3 = 0,$$

$$\bar{\nabla}_{e_1} e_2 = 0, \quad \bar{\nabla}_{e_2} e_2 = -(1 + \varepsilon)e_3, \quad \bar{\nabla}_{e_3} e_2 = 0,$$

$$\bar{\nabla}_{e_1} e_1 = -(1 + \varepsilon)e_3, \quad \bar{\nabla}_{e_2} e_1 = 0, \quad \bar{\nabla}_{e_3} e_1 = 0.$$

Then the Riemannian and the Ricci curvature tensor \bar{R} , Ricci tensor \bar{S} and scalar curvature $\bar{\tau}$ with respect to the semi-symmetric metric connection are given by:

$$\bar{R}(e_1, e_2)e_2 = -(1 + \varepsilon)^2 e_1, \quad \bar{R}(e_1, e_3)e_3 = -\varepsilon(1 + \varepsilon)e_2, \quad \bar{R}(e_2, e_1)e_1 = -(1 + \varepsilon)^2 e_2,$$

$$\bar{R}(e_2, e_3)e_3 = -\varepsilon(1 + \varepsilon)e_2, \quad \bar{R}(e_3, e_1)e_1 = \varepsilon(1 + \varepsilon)e_3, \quad \bar{R}(e_3, e_2)e_2 = -\varepsilon(1 + \varepsilon)e_3,$$

$$\bar{S}(e_1, e_1) = \bar{S}(e_2, e_2) = -(1 + \varepsilon)(1 + 2\varepsilon), \quad \bar{S}(e_3, e_3) = 2\varepsilon(1 + \varepsilon).$$

$$\bar{\tau} = -(1 + \varepsilon).$$

Acknowledgments

The authors are thankful to the referees for their valuable comments and suggestions towards the improvement of the paper.

REFERENCES

- [1] De, U. C. and Sarkar, A., On ε -Kenmotsu manifold, *Hardonic J.* 32 (2009), no.2, 231-242.
- [2] Decu, S., Haesen, S. and Verstraelen, L., Optimal inequalities involving Casorati curvatures, *Bull. Transilv. Univ. Brasov Ser. B.*, 14(49), suppl., (2007), 85-93.
- [3] Bartolotti, E., Sulla geometria della variata a connection affine. *Ann. di Mat.* 4(8) (1930), 53-101.
- [4] Bejancu A. and Duggal K. L., Real hypersurfaces of indefinite Kaehler manifolds, *Int. J. Math. Math. Sci.* 16(1993), no. 3, 545-556.
- [5] Blair, D. E., *Contact manifolds in Riemannian geometry*, Lecture note in Mathematics, 509, Springer-Verlag Berlin-New York, 1976.
- [6] Friedmann, A. and Schouten, J. A., Uber die Geometric der halbsymmetrischen Ubertragung, *Math. Z.* 21 (1924), 211-223.
- [7] Hayden, H. A., Subspaces of space with torsion, *Proc. London Math. Soc.* 34 (1932), 27-50.
- [8] Hiričă, I. E. and Nicolescu, L., Conformal connections on Lyra manifolds, *Balkan J. Geom. Appl.*, 13 (2008), 43-49.
- [9] Hiričă, I. E. and Nicolescu, L., On Weyl structures, *Rend. Circ. Mat. Palermo, Serie II, Tomo LIII*, (2004), 390-400.
- [10] Ahmad M., Rahman S., Siddiqi M. D., Semi-invariant submanifolds of a nearly Sasakian manifold endowed with a semi-symmetric metric connection, *Bull. Allahabad Math. Soc.*, 25, part 1, 23-33, 2010.
- [11] Ahmad M., Jun J. B. and Siddiqi M. D., On some properties of semi-invariant submanifolds of a nearly trans-Sasakian manifolds admitting a quarter-symmetric non-metric connection, *Journal of the Chungcheong Math. Soc.*, Vol. 25, no. 1, 2012.
- [12] Ahmad M., Siddiqi M. D., J. P. Ojha, Semi-invariant submanifolds of a Kenmotsu manifold immersed in a generalized almost r -contact structure admitting a quarter-symmetric non-metric connection, *J. Math. Comput. Sci.* 2(4) 982-998, 2012.
- [13] Haseeb, A., Khan, M. A. and Siddiqi. M. D., Some more results on an ε -kenmotsu manifold with a semi-symmetric metric connection, *Acta Math. Univ. Comenianae*, Vol. LXXXV, 1 (2016), 9-20.
- [14] Jun, J. B., De, U. C. and Pathak, G., On Kenmotsu manifolds, *J. Korean Math. Soc.* 42 (2005), no. 3, 435-445.
- [15] Lee, C. W., Yoon, D. W. and Lee, J. W., Optimal inequalities for the Casorati curvatures of submanifolds of real space forms endowed with semi-symmetric metric connections. *J. Inequal. Appl.* 2014, Article ID 327 (2014).
- [16] Oprea, T., Optimization methods on Riemannian submanifolds, *An. Univ. Bucur., Mat.*, 54 (2005), 127-136.
- [17] Oprea, T., Chen's inequality in the Lagrangian case, *Colloq. Math.*, 108 (2007), 163-169.
- [18] Oprea, T., Ricci curvature of Lagrangian submanifolds in complex space forms, *Math. Inequal. Appl.*, 13(4) (2010), 851-858.
- [19] Siddiqui, A. N., Upper bound inequalities for δ -Casorati curvatures of submanifolds in generalized Sasakian space forms admitting a semi-symmetric metric connection, *Inter. Elec. J. Geom.*, 11(1) (2018), 57-67.
- [20] Kenmotsu, K., A class of almost contact Riemannian manifold, *Tohoku Math. J.*, 24 (1972), 93-103.
- [21] Pathak, G. and De, U. C., On a semi-symmetric connection in a Kenmotsu manifold, *Bull. Calcutta Math. Soc.* 94 (2002), no. 4, 319-324.
- [22] Sharfuddin, A. and Hussain, S. I., Semi-symmetric metric connections in almost contact manifolds, *Tensor (N.S.)*, 30(1976), 133-139.
- [23] Tripathi, M. M., On a semi-symmetric metric connection in a Kenmotsu manifold, *J. Pure Math.* 16(1999), 67-71.
- [24] Tripathi, M. M., Kilic, E., Perktas S. Y. and Keles, S., Indefinite almost para-contact metric manifolds, *Int. J. Math. and Math. Sci.* (2010), art. id 846195, pp. 19.
- [25] Xufeng, X. and Xiaoli, C., Two theorem on ε -Sasakian manifolds, *Int. J. Math. Math. Sci.* 21 (1998), no. 2, 249-54.
- [26] Yano, K., On semi-symmetric metric connections, *Revue Roumaine De Math. Pures Appl.* 15(1970), 1579-1586.
- [27] Yano, K. and Kon, M., *Structures on Manifolds*, Series in Pure Math., Vol. 3, World Sci., 1984.

- [28] Zhang, P. and Zhang, L., Remarks on inequalities for the Casorati curvatures of slant submanifolds in quaternionic space forms, *J Ineq. App.*, 2014, 2014:452.

(Mohammed Danish Siddiqi) JAZAN UNIVERSITY, JAZAN, KINGDOM OF SAUDI ARABIA.

Email address: msiddiqi@jazanu.edu.sa

(Ghodratallah Fasihi-Ramandi) DEPARTMENT OF PURE MATHEMATICS, FACULTY OF SCIENCE, IMAM KHOMEINI INTERNATIONAL UNIVERSITY, QAZVIN, IRAN.

Email address: fasihi@sci.ikiu.ac.ir

(Mohammed Hasan) JAZAN UNIVERSITY, JAZAN, KINGDOM OF SAUDI ARABIA.

Email address: mhhusain@jazanu.edu.sa