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Research Paper

A NOTE ON LOCAL ENTROPY OF RANDOM DYNAMICAL SYSTEMS

MEHDI RAHIMI*, AHMAD SHAKOURI, AND MOHAMMAD MOHAMMADI ANJEDANI

ABSTRACT. In this paper, we review some properties of the entropy of random dynamical systems. We define a local entropy map for random dynamical systems and study some of its properties. We extract the entropy of random dynamical systems from the introduced map.

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Keywords: Entropy, local entropy, random dynamical systems.

1. Introduction

Random dynamical systems, abbreviated by RDS, are generalizations of deterministic dynamical systems, in the sense that, at each time step a transformation is chosen randomly from a given family according to some probability distribution [1]. Indeed, an RDS is characterized by a state space X, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with a usually $\mathbb{P}$-ergodic map $\theta : \Omega \to \Omega$, for which the assignment $\omega \longmapsto \varphi_\omega$ is done randomly.

Many subjects in the ergodic theory of deterministic dynamical systems are discussed for RDSs. Invariant measures, ergodic measures, random ergodic theorems and entropy of random dynamical systems are among the concepts which are discussed for RDSs [2, 31, 11]. These concepts are mainly motivated from deterministic dynamical systems.

Local approaches to the entropy of dynamical systems are discussed and studied extensively [30, 18, 5, 6, 29, 20, 16]. Motivated by localization of entropy in deterministic dynamical systems [25, 26, 27], we present a local view to the entropy of RDSs. To do this, we introduce a map which is defined at each point of the space $\Omega \times X$ and is nearly related to the entropy of the RDS defined on $\Omega \times X$, in the sense that the entropy of an RDS may be extracted by integration of the introduced map.

In Section 2, we provide some preliminary concepts and backgrounds which are necessary for the rest of the paper. In Section 3, we present a local approach to the entropy of RDSs. We conclude the paper with a concluding remark.

2. Preliminary facts

In this section, we review some preliminary concepts and facts. The definitions and concepts discussed in this section are mainly from [7, 8]. One may see [1, 11, 12, 14] for general theory of RDS. Let X be a compact metric space with the Borel σ-algebra $\mathcal{B}_X$. We also write

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*Corresponding author.

$C(X,X)$ for the set of continuous functions $f : X \to X$ with C^0-topology. Let also $(\Omega, \mathcal{F}, \mathbb{P})$ be a countably generated probability space, and $\theta : \Omega \to \Omega$ an invertible $\mathbb{P}$-ergodic system.

A measurable map $\varphi : (\Omega, \mathcal{F}, \mathbb{P}) \to C(X,X)$ defined by $\omega \mapsto \varphi_\omega$ is called a random dynamical system on X over $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$.

The skew product transformation associated with a random dynamical system φ is defined by

$$\Phi : \Omega \times X \to \Omega \times X$$
$$(\omega, x) \mapsto (\theta(\omega), \varphi_\omega(x)).$$

For $n \geq 1$ we also write

$$\varphi_\omega^n := \begin{cases} id_X & n = 0 \\ \varphi_{\theta^{n-1}(\omega)} \circ \ldots \circ \varphi_{\theta(\omega)} \circ \varphi_\omega & n \geq 1 \end{cases}$$

If $\pi_\Omega : \Omega \times X \to \Omega$ is the projection on Ω, we easily have $\pi_\Omega \circ \Phi = \theta \circ \pi_\Omega$.

A probability measure μ on $(\Omega \times X, \mathcal{F} \times \mathcal{B}_X)$ is said to be φ-invariant if

(1) μ is invariant under Φ.
(2) $\pi_\Omega^* \mu = \mathbb{P}$, where $\pi_\Omega^* \mu : \mathcal{F} \to [0,1]$ defined by $\pi_\Omega^* \mu(\Lambda) = \mu(\pi_\Omega^{-1}(\Lambda))$ is called the marginal of μ.

We denote the collection of φ-invariant measures by $I_{\mathbb{P}}(\varphi)$.

An invariant measure μ on $\mathcal{F} \times \mathcal{B}_X$ is called φ-ergodic if μ is Φ-ergodic. The collection of all φ-ergodic measures is denoted by $E_{\mathbb{P}}(\varphi)$.

2.1. **Disintegration of a measure.** In the rest of the paper, we assume that μ disintegrates with respect to $\mathbb{P}$, i.e, there is a family of conditional probability measures $\{\mu_\omega\}_{\omega \in \Omega}$ on $\mathcal{B}_X$ such that $d\mu(\omega, x) = d\mu_\omega(x) d\mathbb{P}(\omega)$. Indeed, If X is a Polish space, then any φ-invariant measure μ has a unique disintegration [1].

Note that, the previous condition is equivalent to

$$\mu(D) = \int_\Omega \int_X \chi_D(\omega, x) d\mu(x) d\mathbb{P}(\omega),$$

for $D \in \mathcal{F} \times \mathcal{B}_X$.

Also, the condition $d\mu(\omega, x) = d\mu_\omega(x) d\mathbb{P}(\omega)$ results in the following:

(1) $\int_{\Omega \times X} f d\mu = \int_\Omega \int_X f(\omega, x) d\mu_\omega(x) d\mathbb{P}(\omega)$ for all $f \in L^1(\mu)$.
(2) If $A \in \mathcal{F} \times \mathcal{B}_X$ and $\omega \in \Omega$, then

$$\mu(A) = \int_\Omega \mu_\omega(A_\omega) d\mathbb{P}(\omega),$$

where $A_\omega = \{x \in X : (\omega, x) \in A\}$ is the ω-section of A.

2.2. **Weak* topology.** Denote $\|f\|_b := \sup_{x \in X} |f(x)|$ for any $f \in C(X)$. A function $f : \Omega \to C(X)$, $\omega \mapsto f_\omega = f(\omega, .)$ is called measurable if the function $(\omega, x) \mapsto f_\omega(x) = f(\omega, x)$ is measurable. The set of all measurable functions $f : \Omega \to C(X)$ with $\|f\| := \int_\Omega \|f_\omega\|_b d\mathbb{P}(\omega) < +\infty$ is denoted by $L^1_{\mathbb{P}}(\Omega, C(X))$. It is easily seen that $L^1_{\mathbb{P}}(\Omega, C(X))$ is a Banach space.

Let $\mathcal{M}(X)$ be the set of all complex Borel measures on X, equipped by the norm $\| \cdot \| = | \cdot |(X)$. Now let $L^\infty_{\mathbb{P}}(\Omega, \mathcal{M}(X))$ be the set of functions $\mu : \Omega \to \mathcal{M}(X)$, $\omega \mapsto \mu_\omega$ with $||\mu||_\infty < +\infty$ where

$$\|\mu\|_\infty := \inf\{M > 0 : \|\mu_\omega\| = |\mu_\omega|(X) < M, \quad \text{for} \quad \mathbb{P}.a.e.\ \omega \text{ in } \Omega\}$$

Since $\mathcal{M}(X)$ is the dual of $C(X)$ we will have:

$$L^1_{\mathbb{P}}(\Omega, C(X))^* = L^\infty_{\mathbb{P}}(\Omega, \mathcal{M}(X)).$$

This induces the weak* topology on $I_{\mathbb{P}}(\mu) \subset L^\infty_{\mathbb{P}}(\Omega, \mathcal{M}(X))$. Indeed, for $\{\mu_n\}_{n\geq 1}$ and μ in $I_{\mathbb{P}}(\mu)$ we have:

$$\mu_n \to \mu \Longleftrightarrow \forall f \in L^1_{\mathbb{P}}(\Omega, C(X)) \quad \int_{\Omega\times X} f d\mu_n \to \int_{\Omega\times X} f d\mu.$$

In light of Krein-Milman theorem, we have the following proposition.

Proposition 2.1. *Let X be a compact metric space and φ be a continuous random dynamical system over $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$. Then*

(1) $I_{\mathbb{P}}(\varphi)$ is a non-empty convex compact subset of $L^\infty_{\mathbb{P}}(\Omega, \mathcal{M}(X))$.
(2) The set of extreme points of $I_{\mathbb{P}}(\varphi)$ is equal to $E_{\mathbb{P}}(\varphi)$.

For a compact metric space X, $C(X)$ is separable. Since $(\Omega, \mathcal{F}, \mathbb{P})$ is countably generated, $L^1_{\mathbb{P}}(\Omega, C(X))$ is also separable, so the topology of weak convergence of the compact Hausdorff space $L^\infty_{\mathbb{P}}(\Omega, \mathcal{M}(X))$ will be metrizable. Therefore, applying the Choquet's theorem [19] to $I_{\mathbb{P}}(\varphi)$, we have the following ergodic decomposition for the elements of $I_{\mathbb{P}}(\varphi)$.

Proposition 2.2. *Suppose that $\mathbb{P}$ is θ-ergodic. Then for each $\mu \in I_{\mathbb{P}}(\varphi)$ there exists a unique Borel probability measure $\tau = \tau_\mu$ on the σ-algebra of Borel subsets of $I_{\mathbb{P}}(\varphi)$ with $\tau_\mu(E_{\mathbb{P}}(\varphi)) = 1$ and that*

$$\int_{\Omega\times X} f d\mu = \int_{E_{\mathbb{P}}(\varphi)} \left(\int_{\Omega\times X} f d\nu \right) d\tau_\mu(\psi)$$

for any $f \in L^1_{\mathbb{P}}(\Omega, C(X))$.

Under the assumptions of the previous proposition, we write $\mu = \int_{E_{\mathbb{P}}(\varphi)} \nu d\tau_\mu(\nu)$ and call it the ergodic decomposition of μ.

2.3. **The entropy of random dynamics.** The entropy of dynamical systems was first defined in [13, 32], and then was studied from other view points in [10, 17, 20, 28, 29].

Using the ideas in classical dynamical systems, this quantity is formulated for random dynamical systems [7, 8, 35, 34].

Let ξ be a finite measurable partition of $\Omega \times X$. For $\omega \in \Omega$, set $\xi_\omega = \{D_\omega\}_{\omega\in\Omega}$, where D_ω is the ω-section of D. Clearly, ξ_ω is a finite partition of X.

For $\omega \in \Omega$, set

$$H_{\mu_\omega}(\bigvee_{i=0}^{n-1} (\varphi^i_\omega)^{-1}\xi_{\theta^i(\omega)}) := -\sum_{A\in\bigvee_{i=0}^{n-1}(\varphi^i_\omega)^{-1}\xi_{\theta^i(\omega)}} \mu_\omega(A)\log\mu_\omega(A)$$

and

$$h^r_\mu(\varphi, \xi) := \lim_{n\to\infty} \frac{1}{n} \int_\Omega H_{\mu_\omega}(\bigvee_{i=0}^{n-1} (\varphi^i_\omega)^{-1}\xi_{\theta^i(\omega)}) d\mathbb{P}(\omega). \tag{2.1}$$

Note that, the limit in (2.1) exists. Also, if $\mathbb{P}$ is θ-ergodic, then

$$h^r_\mu(\varphi, \xi) = \lim_{n\to\infty} \frac{1}{n} H_{\mu_\omega}(\bigvee_{i=0}^{n-1} (\varphi^i_\omega)^{-1}\xi_{\theta^i(\omega)}) \tag{2.2}$$

for $\mathbb{P}.a.e.\omega$ in Ω.

Finally, the entropy of the random dynamical system φ is defined as $h^r_\mu(\varphi) = \sup_\xi h^r_\mu(\varphi, \xi)$ where the supremum is taken over all finite measurable partitions of $\Omega \times X$. We also have the following lemma.

Lemma 2.3. *Given any partition ξ of $\Omega \times X$, the mapping $\mu \mapsto h^r_\mu(\varphi, \xi)$ is affine.*

Proof. Let $\mu, \nu \in I_{\mathbb{P}}(\varphi)$ and $\lambda \in [0, 1]$. First note that, if $\{\mu_\omega\}_{\omega\in\Omega}$ and $\{\nu_\omega\}_{\omega\in\Omega}$ are disintegrations of μ and ν respectively, then $\{\lambda\mu_\omega + (1-\lambda)\nu_\omega\}_{\omega\in\Omega}$ is a disintegration of $\lambda\mu + (1-\lambda)\nu$. On the other hand, for $\omega \in \Omega$ and $n \geq 1$, since the function $\eta(s) = -s \log s$ is concave and $\gamma(s) = \log s$ is increasing, we have:

$$0 \leq H_{\lambda\mu_\omega+(1-\lambda)\nu_\omega}(\xi^n_\omega) - \lambda H_{\mu_\omega}(\xi^n_\omega) - (1-\lambda)H_{\nu_\omega}(\xi^n_\omega) \leq \log 2$$

where $\xi^n_\omega := \bigvee_{i=0}^{n-1}(\varphi^i_\omega)^{-1}\xi_{\theta^i(\omega)}$. Finally, the previous inequalities easily result in

$$h^r_{\lambda\mu+(1-\lambda)\nu}(\varphi, \xi) = \lambda h^r_\mu(\varphi, \xi) + (1-\lambda)h^r_\nu(\varphi, \xi)$$

which completes the proof. □

At the end of this section, we review the Abramov-Rokhlin theorem which connects the entropy of a random dynamical system $h^r_\mu(\varphi)$ to the classical Kolmogrov entropies $h_\mu(\Phi)$ and $h_{\mathbb{P}}(\theta)$.

Theorem 2.4. [3] *Let $\varphi : (\Omega, \mathcal{F}, \mathbb{P}) \to C(X, X)$ be a random dynamical system on X over $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ and Φ be the corresponding skew product. Then, we have*

$$h_\mu(\Phi) = h^r_\mu(\varphi) + h_{\mathbb{P}}(\theta).$$

3. Local Approach

In this section, we present a local approach to the entropy of random dynamical systems. It obviously results in a local entropy for classical dynamical systems as a special case.

In the rest of paper, X is a compact metric space, $\varphi : (\Omega, \mathcal{F}, \mathbb{P}) \to C(X, X)$ is a continuous random dynamical system on X over $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ and Φ is the corresponding skew product.

Definition 3.1. For $\omega \in \Omega, x \in X$ and $D \in \mathcal{F} \times \mathcal{B}_X$, set

$$\gamma_\varphi(\omega, x, D) := \limsup_{n\to\infty} \frac{1}{n} \operatorname{card}(\{0 \leq j \leq n-1 : \varphi^j_\omega(x) \in D_{\theta^j(\omega)}\}).$$

Definition 3.2. Let $\omega \in \Omega, x \in X$and ξ be a finite measurable partition of $\Omega \times X$. Let $g : [0, 1] \to \mathbb{R}$ be the function defined by $g(0) = 0$ and $g(s) = -s \log s$ $(s \in (0, 1])$. Set

$$\Gamma_\varphi(\omega, x; \xi) := \sum_{D\in\xi} g(\gamma_\varphi(\omega, x, D)). \tag{3.1}$$

Definition 3.3. The local entropy map of the random dynamical system φ over $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ with respect to the partition ξ, is a map $\mathcal{J}^r_\varphi(\cdot, \cdot; \xi) : \Omega \times X \to \mathbb{R}$ defined by

$$\mathcal{J}^r_\varphi(\omega, x; \xi) := \mathcal{J}_\varphi(\omega, x; \xi) - h_{\mathbb{P}}(\theta),$$

where

$$\mathcal{J}_\varphi(\omega, x; \xi) := \limsup_{n\to\infty} \frac{1}{n}\Gamma_\varphi(\omega, x; \bigvee_{i=0}^{n-1}(\Phi^i)^{-1}\xi).$$

The conditional version of (3.1) is also defined as follows:

$$\Gamma_\varphi(\omega, x; \xi|\eta) := - \sum_{A\in\xi, B\in\eta} \gamma_\varphi(\omega, x, A\cap B) \log \frac{\gamma_\varphi(\omega, x, A\cap B)}{\gamma_\varphi(\omega, x, B)}.$$

The following proposition states some of the properties of the previous quantities.

Proposition 3.4. *Let $\omega \in \Omega$, $x \in X$ and ξ, η and ζ be finite measurable partitions of $\Omega \times X$. Then,*

(1) $0 \leq \Gamma_\varphi(\omega, x; \xi|\zeta) \leq \Gamma_\varphi(\omega, x; \xi \vee \eta|\zeta)$.
(2) $\Gamma_\varphi(\omega, x; \xi \vee \eta|\zeta) \geq \Gamma_\varphi(\omega, x; \xi|\zeta) + \Gamma_\varphi(\omega, x; \eta|\xi \vee \zeta)$.
(3) If $\xi < \eta$ *then* $\Gamma_\varphi(\omega, x; \xi|\zeta) \leq \Gamma_\varphi(\omega, x; \eta|\zeta)$.

Proof.

(1) First note that, for $\omega \in \Omega$ and $x \in X$, we have:

$$\sum_{B\in\eta} \gamma_\varphi(\omega, x, A\cap B\cap C) \geq \gamma_\varphi(\omega, x, A\cap C). \tag{3.2}$$

Applying (3.2), we will have:

$$\begin{aligned}
\Gamma_\varphi(\omega, x; \xi \vee \eta|\zeta) &= - \sum_{A\in\xi, B\in\eta, C\in\zeta} \gamma_\varphi(\omega, x, A\cap B\cap C) \log \frac{\gamma_\varphi(\omega, x, A\cap B\cap C)}{\gamma_\varphi(\omega, x, C)} \\
&\geq - \sum_{A\in\xi, B\in\eta, C\in\zeta} \gamma_\varphi(\omega, x, A\cap B\cap C) \log \frac{\gamma_\varphi(\omega, x, A\cap C)}{\gamma_\varphi(\omega, x, C)} \\
&\geq - \sum_{A\in\xi, C\in\zeta} \gamma_\varphi(\omega, x, A\cap C) \log \frac{\gamma_\varphi(\omega, x, A\cap C)}{\gamma_\varphi(\omega, x, C)} \\
&= \Gamma_\varphi(\omega, x; \xi|\zeta).
\end{aligned}$$

(2) Let ξ, η and ζ be finite measurable partition of $\Omega \times X$. Without loss of generality, we may assume that $\gamma_\varphi(\omega, x, B) > 0$ for all sets in ξ, η and ζ. Now, we have:

$$\Gamma_\varphi(\omega, x; \xi \vee \eta|\xi) = - \sum_{A\in\xi, B\in\eta, C\in\zeta} \gamma_\varphi(\omega, x, A\cap B\cap C) \log \frac{\gamma_\varphi(\omega, x, A\cap B\cap C)}{\gamma_\varphi(\omega, x, C)}.$$

Since

$$\frac{\gamma_\varphi(\omega, x, A\cap B\cap C)}{\gamma_\varphi(\omega, x, C)} = \frac{\gamma_\varphi(\omega, x, A\cap B\cap C)}{\gamma_\varphi(\omega, x, A\cap C)} . \frac{\gamma_\varphi(\omega, x, A\cap C)}{\gamma_\varphi(\omega, x, C)}$$

(Note that, if $\gamma_\varphi(\omega, x, A\cap C) = 0$, the left hand side is zero and we need not consider it), then

$$\begin{aligned}\Gamma_\varphi(\omega, x; \xi\vee\eta|\zeta) &= -\sum_{A\in\xi, B\in\eta, C\in\zeta}\gamma_\varphi(\omega, x, A\cap B\cap C)\log\frac{\gamma_\varphi(\omega, x, A\cap C)}{\gamma_\varphi(\omega, x, C)}\\ &\quad -\sum_{A\in\xi, B\in\eta, C\in\zeta}\gamma_\varphi(\omega, x, A\cap B\cap C)\log\frac{\gamma_\varphi(\omega, x, A\cap B\cap C)}{\gamma_\varphi(\omega, x, A\cap C)}\\ &\geq -\sum_{A\in\xi, C\in\zeta}\gamma_\varphi(\omega, x, A\cap C)\log\frac{\gamma_\varphi(\omega, x, A\cap C)}{\gamma_\varphi(\omega, x, C)} + \Gamma_\varphi(\omega, x; \eta|\xi\vee\zeta)\\ &= \Gamma_\varphi(\omega, x; \xi|\zeta) + \Gamma_\varphi(\omega, x; \eta|\xi\vee\zeta).\end{aligned}$$

(3) Since $\xi < \eta$ then $\xi\vee\eta = \eta$. Therefore, applying part (ii),

$$\Gamma_\varphi(\omega, x; \eta|\zeta) = \Gamma_\varphi(\omega, x; \xi\vee\eta|\zeta) \geq \Gamma_\varphi(\omega, x; \xi|\zeta) + \Gamma_\varphi(\omega, x; \eta|\xi\vee\zeta) \geq \Gamma_\varphi(\omega, x; \xi|\zeta). \quad \square$$

In the theory of deterministic dynamical systems, a measure-theoretic dynamical system $(Y, \mathcal{B}, \nu, g)$ is said to be a factor of $(X, \mathcal{A}, \mu, f)$ if there is a measure preserving map $\pi : (X, \mathcal{A}, \mu) \to (Y, \mathcal{B}, \nu)$ such that the following diagram is commutative μ-almost everywhere:

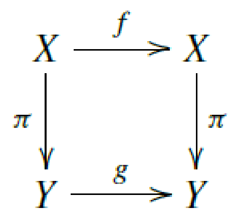

Figure 1

It is proved that, whenever $(X, \mathcal{A}, \mu, f)$ and $(Y, \mathcal{B}, \nu, g)$ are Lebesgue spaces, if g is a factor of f via the measurable map $\pi : X \to Y$ such that card$(\pi^{-1}\{y\})$ is finite for ν-almost every y in Y then $h_\mu(f) = h_\nu(g)$ [15].

A similar result is formulated and proved for random dynamical systems in [15]. We formulate and state a local version of the invariance of entropy for factors.

Proposition 3.5. *Let $\varphi : (\Omega, \mathcal{F}, \mathbb{P}) \to C(X, X)$ defined by $\omega \longmapsto \varphi_\omega$ and $\psi : (\Omega, \mathcal{F}, \mathbb{P}) \to C(Y, Y)$ defined by $\omega \longmapsto \psi_\omega$ be two random dynamical systems over $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$. Assume that there exists a family of measurable maps $\{\pi_\omega : X \to Y\}_{\omega\in\Omega}$ such that, for every $\omega \in \Omega$, the following diagram is commutative:*

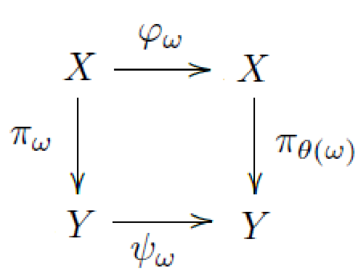

Figure 2

where $\Pi : \Omega \times X \to \Omega \times Y$, $(\omega, x) \longmapsto (\omega, \pi_\omega(x))$ *is a measurable map. Then, for every finite measurable partition* ξ *of* $\Omega \times Y$, *we have*

$$\mathcal{J}^r_\psi(\cdot, \cdot; \xi) \circ \Pi = \mathcal{J}^r_\varphi(\cdot, \cdot; \Pi^{-1}(\xi)),$$

or equivalently, the following diagram is commutative:

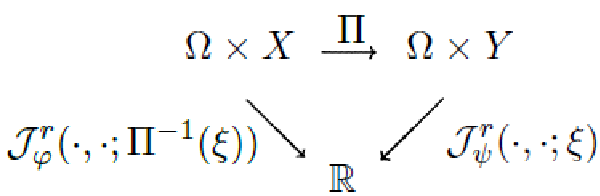

Figure 3

Proof. Let Φ and Ψ be the corresponding skew products of φ and ψ respectively. Since the diagram in Figure 2 is commutative, we have $\Psi \circ \Pi = \Pi \circ \Phi$. Let $D \in \mathcal{F} \times \mathcal{B}_Y$ and $(\omega, x) \in \Omega \times X$. We have

$$\begin{aligned}
\gamma_\psi(\omega, \pi_\omega(x), D) &= \limsup_{n \to +\infty} \frac{1}{n} \operatorname{card}(\{0 \le j \le n-1 : \ \psi^j_\omega(\pi_\omega(x)) \in D_{\theta^j(\omega)}\}) \\
&= \limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_D(\Psi^j(\omega, \pi_\omega(x))) \\
&= \limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_D((\Psi^j \circ \Pi)(\omega, x)) \\
&= \limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_D((\Pi \circ \Phi^j)(\omega, x)) \\
&= \limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_{\Pi^{-1}(D)}((\Phi^j)(\omega, x)) \\
&= \gamma_\varphi(\omega, x, \Pi^{-1}(D)).
\end{aligned}$$

Therefore, given any measurable partition ξ of $\Omega \times Y$, we conclude that

$$\Gamma_\psi(\omega, \pi_\omega(x); \xi) = \Gamma_\varphi(\omega, x; \Pi^{-1}(\xi)). \tag{3.3}$$

Replacing ξ by $\bigvee_{j=0}^{n-1}\Psi^{-j}(\xi)$ in (3.3), we will have

$$\begin{aligned}
\mathcal{J}_\psi(\omega,\pi_\omega(x);\xi) &= \limsup_{n\to+\infty}\frac{1}{n}\Gamma_\psi(\omega,\pi_\omega(x);\bigvee_{j=0}^{n-1}\Psi^{-j}(\xi)) \\
&= \limsup_{n\to+\infty}\frac{1}{n}\Gamma_\varphi(\omega,x;\Pi^{-1}(\bigvee_{j=0}^{n-1}\Psi^{-j}(\xi)) \\
&= \limsup_{n\to+\infty}\frac{1}{n}\Gamma_\varphi(\omega,x;\bigvee_{j=0}^{n-1}\Pi^{-1}(\Psi^{-j}(\xi)) \\
&= \limsup_{n\to+\infty}\frac{1}{n}\Gamma_\varphi(\omega,x;\bigvee_{j=0}^{n-1}\Phi^{-j}(\Pi^{-1}(\xi)) \\
&= \mathcal{J}_\varphi(\omega,x;\Pi^{-1}(\xi)).
\end{aligned}$$

This completes the proof. □

Now, we are ready to show that the map $\mathcal{J}_\varphi^r(\cdot,\cdot;\xi)$ is indeed a local entropy map.

Theorem 3.6. *For any $\mu\in I_{\mathbb{P}}(\varphi)$ we have:*

$$h_\mu^r(\varphi)=\sup_\xi\int_{\Omega\times X}\mathcal{J}_\varphi^r(\omega,x;\xi)d\mu(\omega,x)$$

where the supremum is taken over all finite measurable partitions of $\Omega\times X$.

Proof. First, let $\mu\in E_{\mathbb{P}}(\varphi)$. Let $A\in\mathcal{F}\times\mathcal{B}_X$, then by Birkhoff ergodic theorem,

$$\begin{aligned}
\gamma_\varphi(\omega,x,A) &= \limsup_{n\to\infty}\frac{1}{n}\operatorname{card}(\{0\le j\le n-1:\varphi_\omega^j(x)\in A_{\theta^j(\omega)}\}) \\
&= \limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\chi_{A_{\theta^j(\omega)}}(\varphi_\omega^j(x)) \\
&= \limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\chi_A(\theta^j(\omega),\varphi^j(x)) \\
&= \limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\chi_A(\Phi^j(\omega,x)) \\
&= \int_{\Omega\times X}\chi_A d\mu=\mu(A) \qquad (3.4)
\end{aligned}$$

for $\mathbb{P}.a.e.\ \omega\in\Omega$.

For any finite measurable partition ξ of $\Omega\times X$, (3.4) results in

$$\Gamma_\varphi(\omega,x;\xi)=H_\mu(\xi) \qquad (3.5)$$

for $\mathbb{P}.a.e.\ \omega\in\Omega$.

For $n \geq 1$, replacing ξ by $\bigvee_{i=0}^{n-1}(\Phi^i)^{-1}\xi$ in (3.5), there exist a $\mathbb{P}$-measurable set $\Lambda_n \subset \Omega \times X$ with $\mu(\Lambda_n) = 1$ and

$$\frac{1}{n}\Gamma_\varphi(\omega, x; \bigvee_{i=0}^{n-1}(\Phi^i)^{-1}\xi) = \frac{1}{n}H_\mu(\bigvee_{i=0}^{n-1}(\Phi^i)^{-1}\xi) \tag{3.6}$$

for $(\omega, x) \in \Lambda_n$.

Set $\Lambda := \bigcap_{n=1}^{\infty} \Lambda_n$. Then $\mu(\Lambda) = 1$ and (3.6) holds for any $(\omega, x) \in \Lambda$. Letting $n \to \infty$ in (3.6), we conclude that

$$\mathcal{J}_\varphi(\omega, x; \xi) = h_\mu(\varphi, \xi)$$

for any $(\omega, x) \in \Lambda$. Therefore,

$$\mathcal{J}_\varphi^r(\omega, x; \xi) = h_\mu(\Phi, \xi) - h_{\mathbb{P}}(\theta) \tag{3.7}$$

for any $(\omega, x) \in \Lambda$.

Integrating both sides of (3.7), we will have:

$$\int_{\Omega\times X} \mathcal{J}_\varphi^r(\omega, x; \xi)d\mu(\omega, x) = \int_\Lambda \mathcal{J}_\varphi^r(\omega, x; \xi)d\mu(\omega, x) = h_\mu(\Phi, \xi) - h_{\mathbb{P}}(\theta). \tag{3.8}$$

Now, let in general, $\mu \in I_{\mathbb{P}}(\varphi)$. Let $\mu = \int_{E_{\mathbb{P}}(\varphi)} \nu d\tau(\nu)$ be the ergodic decomposition of μ. Applying (3.8) and Jacob's theorem, we will have:

$$\begin{aligned}
\int_{\Omega\times X} \mathcal{J}_\varphi^r(\omega, x; \xi)d\mu(\nu, x) &= \int_{\Omega\times X} \mathcal{J}_\varphi(\omega, x; \xi)d\mu(\omega, x) - h_{\mathbb{P}}(\theta) \\
&= \int_{E_{\mathbb{P}}(\varphi)} \left(\int_{\Omega\times X} \mathcal{J}_\varphi(\omega, x; \xi)d\nu(\omega, x)\right) d\tau(\nu) - h_{\mathbb{P}}(\theta) \\
&= \int_{E_{\mathbb{P}}(\varphi)} h_\nu(\Phi, \xi)d\tau(\nu) - h_{\mathbb{P}}(\theta) \\
&= h_\mu(\Phi, \xi) - h_{\mathbb{P}}(\theta).
\end{aligned}$$

Finally, the result follows by taking supremume over all measurable partitions ξ of $\Omega \times X$ and the Abramov-Rokhlin theorem. □

4. Discussion and concluding remarks

Local entropies are applied in multifractal analysis to characterize dynamical systems [4, 22, 23, 33]. It studies the dimensional properties of the level sets of certain functions like local entropies, using either Hausdorff dimension or topological entropy in the sense of Bowen [9, 21].

Local entropies may also be applied to measure the amount of information generated by a system in a certain area of the space rather that the whole space. As an example, one may see the definition of information content of a molecular structure via local entropies in [24].

As in the entropy theory of classical dynamical systems, we presented a local view to the concept of entropy of random dynamical systems. We introduced a map $\mathcal{J}_\varphi^r(\cdot, \cdot; \xi) : \Omega \times X \to \mathbb{R}$ which plays the role of a local entropy for the random dynamical system φ over $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$, in the sense that, the entropy of the random dynamical system φ may be extracted by integrating the introduced map $\mathcal{J}_\varphi^r(\cdot, \cdot; \xi)$.

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(Mehdi Rahimi) Department of Mathematics, Faculty of Science, University of Qom, Qom, Iran.
Email address: m10.rahimi@gmail.com

(Ahmad Shakouri) Department of Mathematics, Faculty of Science, University of Qom, Qom, Iran.
Email address: shakourisamad@gmail.com

(Mohammad Mohammadi Anjedani) Department of Mathematics, Faculty of Science, University of Qom, Qom, Iran.
Email address: mm_math67@yahoo.com