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Research Paper

BOUNDEDNESS OF MIKHLIN OPERATOR IN VARIABLE EXPONENT MORREY SPACE

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Abstract. . G. Mikhlin proved the boundedness of the Fourier multiplier operator in the classical Lebesgue space if the multiplier function is a bounded function. In [2], the authors obtained the same result of the classical Morrey space. In this paper, we prove that Mikhlin operator with bounded multiplier function is bounded operator on Morrey space with variable exponent which is containing the classical Lebesgue space with variable exponent and the classical Morrey space.

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1. Introduction and Preliminaries

On 1956 S. G. Mikhlin introduced the generalization of the Fourier multiplier operator and affirmed that it is bounded in Lebesgue space [1]. Fourier multiplier operators play a major role in analysis and in particular in the theory of partial differential equations. For instance [2], the heat diffusion in a wire with length L can be modeled as a partial differential equation. Thus, this problem is extendible to a heat diffusion in a wire with infinite length as follows:

$$u_t = cu_{xx} \quad \forall x \in \mathbb{R},\ t > 0,\ c \in \mathbb{R}$$
$$u(x,0) = \phi(x), \quad x \in \mathbb{R}$$

where $u(x,t)$ is temperature of the wire at some point $x \in \mathbb{R}$ and at the time $t \geq 0$. Hence, one can use the Fourier transformation $\mathcal{F}$ and obtain

$$\hat{u}(x,t) = \mathcal{F}(G * \phi) = (2\pi)^{\frac{n}{2}} \hat{G}\hat{\phi},$$

where $G(x,t) = \frac{1}{\sqrt{4\pi ct}} e^{-\frac{x^2}{4ct}}$. The operator that maps $\hat{\phi}$ to $\hat{u}(x,t)$ is so-called to the multiplier operator generated by $\hat{G}$. In this case, $\hat{G}$ is said to be Fourier multiplier. We are interested in the class of multipliers that satisfy

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the estimates of the standard Mikhlin-Hörmander multiplier theorem which is defined in below [1].

Definition 1.1. Let $N = n+2$, $n \in \mathbb{N}$. Also, suppose that $m : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is an $N-$times differentiable function such that $|\partial_\xi^\alpha m(\xi)| \leq C|\xi|^{-|\alpha|}$ for all $\xi \neq 0$, $|\alpha| \leq N$. A Mikhlin operator is defined as

$$M(f) := \mathcal{F}^{-1}(m\hat{f})$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$.

Theorem 1.2. [1] *Let M be Mikhlin operator. Then M can be extended to a bounded linear operator on $L^p(\mathbb{R}^n)$, for $1 < p < \infty$.*

From Theorem 1.2, using the properties of Fourier transformation in [1], one can obtain Mikhlin operator as follows:

$$M(f)(x) = \int_{\mathbb{R}^n} k(x-y)f(y)dy = (k * f)(x)$$

where function k, so-called the kernel of operator M, satisfies the following properties.

Proposition 1.3. *Suppose that M satisfies Theorem 1.2. Then there is a locally integrable, continuously differentiable function $k : \mathbb{R}^n - \{0\} \to \mathbb{C}$ with compact support which satisfies*

$$|k(z)| \leq C|z|^n \quad |\nabla k(z)| \leq C|z|^{n-1},$$

for all $z \neq 0$ such that $M(f)(x) = (k * f)(x)$ for all $x \notin Supp(f)$, $f \in {}^2(\mathbb{R}^n)$.

The theory of function spaces with variable exponent was extensively studied by researchers since the work of Kov*áč*ik and R*á*kosn*í*k [3] appeared in 1991. Many applications of these spaces were given, for example, in the modeling of electrorheological fluids [4], in the study of image processing [5]. Morrey spaces emerged in close connection with the local behavior of the solutions of elliptic differential equations and they describe local regularity more precisely than Lebesgue spaces; see, for example [6]. The recent survey paper [7]we can find information on various versions of variable exponent Morrey spaces.

Let $p(.)$ be a measurable function on Ω with values in $[1, \infty)$. An open set Ω is assumed to be bounded. Let us consider $1 < p_- \leq p(x) \leq p_+ < \infty$, where $p_- := ess \inf_{x\in\Omega} p(x)$ and $p_+ := ess \sup_{x\in\Omega} p(x)$. We indicate by $L^{p(\cdot)}(\Omega)$ the space of all measurable functions $f(x)$ on Ω such that

$$I_{p(.)}(f) := \int_\Omega |f(x)|^{p(x)}dx < +\infty.$$

This space is equipped with the norm

$$\|f\|_{p(.)} = \inf\left\{\mu > 0 \ : \ I_{p(.)}(\frac{f}{\mu}) \leq 1\right\},$$

which this is a Banach space. Suppose that $p^{'}(x)$ is a conjugate exponent for $p(x)$ i.e., $\frac{1}{p(x)} + \frac{1}{p^{'}(x)} = 1$. The Höder inequality is valid in the from

$$\int_{\Omega} |f(x)||g(x)|dx \leq \left(\frac{1}{p_-} + \frac{1}{p^{'}_-}\right) \|f\|_{p(.)} \|g\|_{p^{'}(.)}.$$

Let $\lambda(x)$ be a measurable function on Ω with values in $[0, n]$. The variable Morrey space $L^{p(.)}_{\lambda(.)}(\Omega)$ is defined as the set of integrable function f on Ω such that

$$\sup_{x \in \Omega, r>0} r^{-\frac{\lambda(x)}{p(x)}} \left(\int\limits_{\Omega \cap B(x,r)} |f(y)|^{p(x)} dy \right) < \infty,$$

for all $y \in \tilde{B}(x, r) := \Omega \cap B(x, r)$. Norm of the functions in this space is

$$\|f\|_{L^{p(x)}_{\lambda(x)}} := \sup_{x \in \Omega, r>0} r^{-\frac{\lambda(x)}{p(x)}} \left(\int\limits_{\Omega \cap B(x,r)} |f(y)|^{p(x)} dy \right)^{\frac{1}{p(x)}}.$$

For the basic on variable exponent Lebesgue spaces we refer to [7, 8] and the references therein.

Definition 1.4. [8] $WL(\Omega)$ (weak Lipschitz) denotes the class of functions defined on Ω satisfying the log-condition

$$(1.1) \qquad |p(x) - p(y)| \leq \frac{A}{-\ln|x-y|}, \quad |x-y| \leq \frac{1}{2}, \; x, y \in \Omega,$$

where $A = A(p) > 0$ does not depend on x, y.

2. Boundedness of Mikhlin operator in Variable Morrey space

Proposition 2.1. *Let M be Mikhlin operator and $p(.) : \Omega \to [1, \infty)$. Then M is a bounded linear operator in the Lebesgue space $L^{p(.)}(\Omega)$.*

Proof. Using of 1.3 and Höder's inequality, and for every $f \in L^{p(x)}(\Omega)$, one gets the following estimations :

$$\begin{aligned}
|M(f)(x)| &= |\int_\Omega k(x-y)f(y)dy| \leq \int_\Omega |k(x-y)||f(y)|dy \\
&\leq C\int_\Omega \frac{|f(y)|}{|x-y|^n}dy \leq C'\int_r^\infty t^{-n-1}\int_{B(x,t)}|f(y)|dydt \\
&\leq C'\int_t^\infty t^{-n-1}\Big(\int_{B(x,t)}|f(y)|^{p(x)}dy\Big)^{\frac{1}{p(x)}}\Big(\int_{B(x,t)}1^{p'(x)}dy\Big)^{\frac{1}{p'(x)}}dt \\
&\leq C'\int_t^\infty t^{-n-1}\Big(\int_{B(x,t)}|f(y)|^{p(x)}dy\Big)^{\frac{1}{p(x)}}t^{\frac{n}{p'(x)}}dt \\
&\leq C''\int_t^\infty t^{-n-1}\Big(\int_\Omega |f(y)|^{p(x)}dy\Big)^{\frac{1}{p(x)}}t^{\frac{n}{p'(x)}}dt \\
&\leq C''r^{\frac{n}{p'(x)}-n}\|f\|_{L^{p(x)}}.
\end{aligned}$$

Indeed,

$$\|M(f)\|_{L^{p(x)}} = \Big(\int_\Omega |M(f)(x)|^{p(x)}dx\Big)^{\frac{1}{p(x)}} \leq C''r^{\frac{n}{p_-}}|\Omega|\|f\|_{L^{p(x)}} < \infty.$$

□

Theorem 2.2. *Let M be Mikhlin operator and $p : \Omega \to [1,\infty)$, $\lambda : \Omega \to [0,n]$ such that $\sup_{x\in\Omega}\lambda(x) < np_+$. Then M can be extended to bounded linear operator on variable Morrey space,i.e.,*

$$M : L^{p(x)}_{\lambda(x)}(\Omega) \to L^{p(x)}_{\lambda(x)}(\Omega).$$

Proof. Suppose that $z \in \Omega$, $r > 0$ and $f \in L^{p(x)}_{\lambda(x)}(\Omega)$. One can write $f = f_1 + f_2$ such that $f_1 = f.\chi_{\tilde{B}(z,2r)}$. As $f \in L^{p(x)}_{\lambda(x)}(\Omega)$ then $f_1 \in L^{p(x)}_{\lambda(x)}(\Omega)$. On the other hand, f_2 is a function in tempered distribution space. One can apply the boundedness of M in $L^{p(x)}$ for which gets:

$$\begin{aligned}
\|M(f_1)\|_{L^{p(x)}(B(z,r))} &\le \|M(f_1)\|_{L^{p(x)}(\Omega)} \le C_1\|f_1\|_{L^{p(x)}(\Omega)} \\
&= C_1\Big(\int\limits_{\Omega} |f_1(x)|^{p(x)}dx\Big)^{\frac{1}{p(x)}} = C_1\Big(\int\limits_{\tilde{B}(z,2r)} |f(z)|^{p(x)}dx\Big)^{\frac{1}{p(x)}} \\
&= C_1 r^{\frac{\lambda(x)}{p(x)}} r^{-\frac{\lambda(x)}{p(x)}}\Big(\int\limits_{\tilde{B}(z,2r)} |f(x)|^{p(x)}dx\Big)^{\frac{1}{p(x)}} \\
&\le C_1 r^{\frac{\lambda(x)}{p(x)}} \sup_{x\in\Omega, r>0} r^{-\frac{\lambda(x)}{p(x)}}\Big(\int\limits_{\tilde{B}(z,2r)} |f(x)|^{p(x)}dx\Big)^{\frac{1}{p(x)}} \\
&\le C_1^{'} r^{np_+}\|f\|_{L^{p(x)}_{\lambda(x)}} \qquad (2.1)
\end{aligned}$$

for all $y \in \tilde{B}(x,2r)$ and for $C_1, C_1^{'} \in \mathbb{R}$. Now, suppose that $z \in \Omega$ and $x \in \tilde{B}(z,r)$. From Proposition 2.1, one obtains

$$\begin{aligned}
|M(f_2)(x)| &= \Big|\int_\Omega k(x-y)f_2(y)dy\Big| \le \int_\Omega |k(x-y)||f_2(y)|dy \\
&\le C_2\int_\Omega \frac{|f_2(y)|}{|x-y|^n}dy = C_2^{'}\int_{\tilde{B}^c(z,2r)} |f(y)|\int_{|x-y|}^{\infty} t^{-n-1}dt, \qquad (2.2)
\end{aligned}$$

for $C_2, C_2^{'} \in \mathbb{R}$. Therefore, by using of Fubini's Theorem, Höder's inequality and the assumption $\sup_{x\in\Omega}\lambda(x) < np_+$, one can conclude the following estimates:

$$\begin{aligned}
|M(f_2)(x)| &\le C_2^{'}\int_r^\infty t^{-n-1}\int_{\tilde{B}(x,t)} |f(y)|dydt \\
&\le C_2^{'}\int_r^\infty t^{-n-1}\Big(\int_{\tilde{B}(x,t)} |f(y)|^{p(x)}dy\Big)^{\frac{1}{p(x)}}\Big(\int_{\tilde{B}(x,t)} 1^{p^{'}(x)}dy\Big)^{\frac{1}{p^{'}(x)}}dt \\
&\le C_2^{'}\int_r^\infty t^{-n-1}\Big(\int_{\tilde{B}(x,t)} |f(y)|^{p(x)}dy\Big)^{\frac{1}{p(x)}} t^{\frac{n}{p^{'}(x)}}dt \\
&\le C_2^{'}\int_r^\infty t^{-n-1+\frac{\lambda(x)}{p(x)}+\frac{n}{p^{'}(x)}} \sup_{x\in\Omega, t>0} t^{-\frac{\lambda(x)}{p(x)}}\Big(\int_{\tilde{B}(x,t)} |f(y)|^{p(x)}dy\Big)^{\frac{1}{p(x)}}dt \\
&\le C_3\|f\|_{L^{p(x)}_{\lambda(x)}(\Omega)}\int_r^\infty t^{-n-1+\frac{\lambda(x)}{p(x)}+\frac{n}{p^{'}(x)}}dt = C_3\|f\|_{L^{p(x)}_{\lambda(x)}} r^{\frac{\lambda(x)-n}{p(x)}},
\end{aligned}$$

for $C_3, C_3^{'} \in \mathbb{R}$. Hence,

$$\|M(f_2)\|_{L^{p(x)}(\tilde{B}(z,r))} = \left(\int_{\tilde{B}(z,r)} M(f_2)(x)|^{p(x)}\right)^{\frac{1}{p(x)}}$$
$$\leq \left(\int_{\tilde{B}(z,r)} |C_3^{'}\|f\|_{L^{p(x)}_{\lambda(x)}} r^{\frac{\lambda(x)-n}{p(x)}}|^{p(x)} dx\right)^{\frac{1}{p(x)}}$$
$$= C_3^{'}\|f\|_{L^{p(x)}_{\lambda(x)}} r^{\frac{\lambda(x)-n}{p(x)}} \|1\|_{L^{p(x)}(\tilde{B}(z,r))} = C_3^{'}\|f\|_{L^{p(x)}_{\lambda(x)}} r^{\frac{\lambda(x)-n}{p(x)}} r^{\frac{n}{p(x)}}$$
$$\leq C_3^{'}\|f\|_{L^{p(x)}_{\lambda(x)}} r^{np_+}. \tag{2.3}$$

According to the relations 2.2 and 2.3, one gets

$$\|M(f)\|_{L^{p(x)}(\tilde{B}(z,r))} = \|M(f_1)\|_{L^{p(x)}(\tilde{B}(z,r))} + \|M(f_2)\|_{L^{p(x)}(\tilde{B}(z,r))}$$
$$C_1^{'}\|f\|_{L^{p(x)}_{\lambda(x)}} r^{np_+} + C_3^{'}\|f\|_{L^{p(x)}_{\lambda(x)}} r^{np_+}$$
$$\leq \max\{C_1^{'}, C_3^{'}\}\|f\|_{L^{p(x)}_{\lambda(x)}} r^{np_+}. \tag{2.4}$$

Since the inequality 2.4 holds for every $r > 0$, thus

$$\|M(f)\|_{L^{p(x)}_{\lambda(x)}(\Omega)} \leq \sup_{z\in\Omega, r>0} r^{-\frac{\lambda(x)}{p(x)}} \|M(f)\|_{L^{p(x)}(\tilde{B}(z,r))} \leq C\|f\|_{L^{p(x)}_{\lambda(x)}(\Omega)}.$$

□

Conclusions

Motivated by the study of maximal and integral operators in harmonic analysis and Electorheological fluids, one can use the boundedness of Mikhlin operators to investigate about the new results of those operators and interesting problems of elasticity, fluid dynamic and calculus of variations with $(p(x), \lambda(x))-$ growth conditions on variable exponents Morrey spaces.

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