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Research Paper

SOME REMARKS ON THE PAPER "GLOBAL OPTIMIZATION IN METRIC SPACES WITH PARTIAL ORDERS

MOOSA GABELEH* AND JACK MARKIN

ABSTRACT. The aim of this note is to show that the main conclusion of a recent paper by Sadiq Basha [S. Sadiq Basha, Global optimization in metric spaces with partial orders, *Optimization, 63 (2014), 817-825*] can be obtained as a consequence of corresponding existing results in fixed point theory in the setting of partially ordered metric spaces. Moreover, by a similar approach, we prove that in the paper [V. Pragadeeswarar, M. Marudai, Best proximity points: approximation and optimization in partially ordered metric spaces, *Optim. Lett. 7 (2013), 1883–1892*] the results are not real generalizations but particular cases of existing fixed point theorems in the literature.

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1. Introduction

Let $(X, \preceq)$ be a partially ordered set. A self mapping $T : X \to X$ is said to be *monotone nondecreasing* if $T(x) \preceq T(y)$ whenever $x, y \in X, x \preceq y$. In 2005 the following fixed point theorem was established by Nieto and Rodri'guez-Lo'pez for monotone nondecreasing mappings which can be considered as an extension of the *Banach contraction principle*. We will provide a brief proof here since the main ideas will be used in the sequel.

Theorem 1.1. ([1]) *Let $(X, \preceq)$ be a partially ordered set and $T : X \to X$ be a self mapping which is monotone nondecreasing. Assume that there is a metric d on X such that (X, d) is a complete metric space and X satisfies the condition*

$$\text{(1.1)} \qquad \textit{if a nondecreasing sequence } \{x_n\} \to x \in X, \textit{ then } x_n \preceq x, \ \forall n.$$

Suppose that there exists $\alpha \in [0, 1[$ such that $d(Tx, Ty) \le \alpha d(x, y)$ for every $x, y \in X$ with $x \preceq y$. If there exists $x_0 \in X$ with $x_0 \preceq T(x_0)$, then T has

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*Corresponding author.

a fixed point. Moreover, if we define $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ converges to a fixed point of T.

Proof. Since $x_0 \in X$ with $x_0 \preceq T(x_0)$ and T is monotone nondecreasing, the Picard's iteration sequence $\{T^n(x_0)\}$ is increasing. It now follows from the assumption on the mapping T that there exists $\alpha \in [0,1[$ such that

$$d(T^{n+1}x_0, T^n x_0) \leq \alpha d(T^n x_0, T^{n-1}x_0), \quad \forall n \in \mathbb{N},$$

that is, $\{T^n(x_0)\}$ is a Cauchy sequence and so converges to an element $p \in X$. By using (1) we conclude that $x_n \preceq p$ for all $n \in \mathbb{N}$. We now have

$$d(T^{n+1}x_0, Tp) \leq \alpha d(T^n x_0, p) \to^{(n\to\infty)} 0,$$

which ensures that p is a fixed point of T. □

Throughout this article we denote by Ψ the class of the *altering distance functions* $\psi : [0,\infty) \to [0,\infty)$ which satisfy the following conditions:
(i) ψ is continuous and nondecreasing;
(ii) $\psi(t) = 0$ if and only if $t = 0$.
This class of functions was first introduced in [6].

In [5] Harjani and Sadarangani established the following extension of Theorem 1.1 by using altering distance functions as control functions on contractive conditions.

Theorem 1.2. ([5]) *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space and X satisfies the condition* (1) *of Theorem 1.1. Let $T : X \to X$ be a monotone nondecreasing self mapping such that*

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \varphi(d(x,y)), \quad \forall x,y \in X \text{ with } x \preceq y, \tag{1.2}$$

where $\psi, \varphi \in \Psi$. If there exists $x_0 \in X$ with $x_0 \preceq T(x_0)$, then T has a fixed point. Moreover, if we define $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ converges to the fixed point of T.

Recently, Theorem 1.1 and Theorem 1.2 was generalized in [9] and [7] in order to resolve an optimization problem in the setting of a metric space that is endowed with a partial order.

In this article we show that the results of [7, 9] not only are not real extensions of Theorem 1.1, Theorem 1.2 but also they are consequences of Theorem 1.1 and Theorem 1.2, respectively. We refer to [3, 4] for more related subject.

2. Preliminaries

Let (X,d) be a metric space equipped with a partial order relation "$\preceq$" and (A,B) be a pair of nonempty subsets of X. We use the following notions and notations in the sequel:

$$\text{dist}(A,B) := \inf\{d(x,y) : (x,y) \in A \times B\},$$
$$A_0 := \{x \in A : d(x,y) = \text{dist}(A,B), \text{ for some } y \in B\},$$

$$B_0 := \{y \in B : d(x,y) = \operatorname{dist}(A,B), \text{ for some } x \in A\},$$

We mention that a point $x^\star \in A$ is said to be a *best proximity point* for a non-self mapping $T : A \to B$ provided that

$$d(x^\star, Tx^\star) = \operatorname{dist}(A,B).$$

It is remarkable to note that if $x^\star \in A$ is a best proximity point for the non-self mapping T, then it is a solution of the following minimization problem: Find

$$\min_{x \in A} d(x, Tx). \tag{2.1}$$

Definition 2.1. ([10]) The pair (A,B) is said to have P-property if and only if

$$\begin{cases} d(x_1,y_1) = \operatorname{dist}(A,B), \\ d(x_2,y_2) = \operatorname{dist}(A,B), \end{cases} \Longrightarrow d(x_1,x_2) = d(y_1,y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Definition 2.2. ([8]) A non-self mapping $T : A \to B$ is said to be proximally increasing if it satisfies the condition that

$$\begin{cases} x_1 \preceq x_2, \\ d(u_1, Tx_1) = \operatorname{dist}(A,B), \\ d(u_2, Tx_2) = \operatorname{dist}(A,B), \end{cases} \Longrightarrow u_1 \preceq u_2,$$

for all $x_1, x_2, u_1, u_2 \in A$.

Definition 2.3. ([8]) A non-self mapping $T : A \to B$ is said to be an ordered proximal contraction if there exists a non-negative real number $\alpha < 1$ such that

$$\begin{cases} x_1 \preceq x_2, \\ d(u_1, Tx_1) = \operatorname{dist}(A,B), \\ d(u_2, Tx_2) = \operatorname{dist}(A,B), \end{cases} \Longrightarrow d(u_1,u_2) \leq \alpha d(x_1,x_2),$$

for all $x_1, x_2, u_1, u_2 \in A$.

Definition 2.4. ([9]) Given non-self mappings $S, T : A \to B$ the pair $(S;T)$ is said to be proximally increasing if

$$\begin{cases} x \preceq y, \\ d(u, Sx) = \operatorname{dist}(A,B), \\ d(v, Ty) = \operatorname{dist}(A,B), \end{cases} \Longrightarrow u \preceq v,$$

for all $x, u \in A$, $y, v \in B$.

Definition 2.5. ([9]) Given non-self mappings $S, T : A \to B$ the pair $(S;T)$ is form an ordered proximal cyclic contraction if there exists a non-negative real number $\beta < 1$ such that

$$\begin{cases} x \preceq y, \\ d(u, Sx) = \mathrm{dist}(A, B), \\ d(v, Ty) = \mathrm{dist}(A, B), \end{cases} \implies d(u, v) \leq \beta d(x, y) + (1 - \beta)\mathrm{dist}(A, B),$$

for all $x, u \in A$, $y, v \in B$.

Here we state the main results of [7, 9].

Theorem 2.6. (see Theorem 3.1 of [9]) *Let X be a nonempty set such that $(X, \preceq)$ is a partially ordered set and (X, d) is a complete metric space. Let A and B be non-void closed subsets of the metric space (X, d) such that A_0 is nonempty. Let $S, T : A \to B$ and $g : A \cup B \to A \cup B$ satisfy the following conditions:*
(i) S and T are proximally increasing, ordered proximal contractions;
(ii) $S(A_0) \subseteq B_0$ and $T(B_0) \subseteq A_0$;
(iii) g is a surjective isometry, its inverse is an increasing mapping, $A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$;
(iv) The pair $(S; T)$ forms a proximally increasing, ordered proximal cyclic contraction.
(v) There exist elements $x_0, x_1 \in A_0$ and $y_0, y_1 \in B_0$ such that

$$d(gx_1, Sx_0) = \mathrm{dist}(A, B) = d(gy_1, Ty_0),$$

where $x_0 \preceq x_1, y_0 \preceq y_1$ and $x_0 \preceq y_0$;
(vi) The sets A and B satisfy the condition (1) of Theorem 1.1.

Then there exists an element $(x^\star, y^\star) \in A \times B$ such that

$$d(gx^\star, Sx^\star) = d(gy^\star, Ty^\star) = d(x^\star, y^\star) = \mathrm{dist}(A, B).$$

Further the sequence $(\{x_n\}, \{y_n\})$ in $A_0 \times B_0$ defined by

$$d(gx_{n+1}, Sx_n) = \mathrm{dist}(A, B) = d(gy_{n+1}, Ty_n), \quad \forall n \in \mathbb{N} \cup \{0\},$$

converges to the element $(x^\star, y^\star)$.

Theorem 2.7. (see Theorems 2.1 and 2.2 of [7]) *Let X be a nonempty set such that $(X, \preceq)$ is a partially ordered set and (X, d) is a complete metric space. Let A and B be non-void closed subsets of the metric space (X, d) such that A_0 is nonempty. Let $T : A \to B$ satisfy the following conditions:*
(i) T is a proximally increasing such that $T(A_0) \subseteq B_0$ and (A, B) satisfies the P-property;
(ii) there exist elements x_0 and x_1 in A_0 such that

$$x_0 \preceq x_1, \quad d(x_1, Tx_0) = \mathrm{dist}(A, B),$$

(iii) for all $x, y \in A$ with $x \preceq y$,

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)), \tag{2.2}$$

where $\varphi, \psi \in \Psi$*;*
(iv) *The set* A *satisfies the condition* (1) *of Theorem 1.1.*

Then T *has a best proximity point. Further the sequence* $\{x_n\}$ *defined by*

$$d(x_{n+1}, Tx_n) = \text{dist}(A, B), \quad \forall n \in \mathbb{N} \cup \{0\},$$

converges to the best proximity point of T.

3. Main results

Theorem 3.1. *Theorem 2.6 is a straightforward consequence of Theorem 1.1.*

Proof. Let $x \in A_0$. Since $Sx \in B_0$, there exists an element $u \in A_0$ such that $d(u, Sx) = \text{dist}(A, B)$. By the fact that $A_0 \subseteq g(A_0)$, we can find an element $\hat{u} \in A_0$ for which $u = g\hat{u}$ and so $d(g\hat{u}, Sx) = \text{dist}(A, B)$. It is worth noticing that if there exists another element $\check{u} \in A_0$ for which $d(g\check{u}, Sx) = \text{dist}(A, B)$, then by this reality that S is an ordered proximal contraction and g is an isometry, we obtain

$$d(\hat{u}, \check{u}) = d(g\hat{u}, g\check{u}) \leq \alpha d(x, x) = 0,$$

which implies that $\hat{u} = \check{u}$. Thus we can define a self mapping $\Pi_1 : A_0 \to A_0$ such that $d(g\Pi_1 x, Sx) = \text{dist}(A, B)$ for all $x \in A_0$. By a similar argument we consider the self mapping $\Pi_2 : B_0 \to B_0$ for which $d(g\Pi_2 y, Ty) = \text{dist}(A, B)$ for any $y \in B_0$. We have the following observations about the mappings Π_i for $i \in \{1, 2\}$.
♣ Let $x_1, x_2 \in A_0$ be such that $x_1 \preceq x_2$. Then

$$\begin{cases} d(g\Pi_1 x_1, Sx_1) = \text{dist}(A, B), \\ d(g\Pi_1 x_2, Sx_2) = \text{dist}(A, B). \end{cases}$$

Since S is a proximally increasing, $g\Pi_1 x_1 \preceq g\Pi_1 x_2$. Since g^{-1} is increasing, we must have $\Pi_1 x_1 \preceq \Pi_1 x_2$, that is, Π_1 is monotone nondecreasing. Equivalently, we can see that Π_2 is also monotone nondecreasing.
♣ Let $x_1, x_2 \in A_0$ be such that $x_1 \preceq x_2$. Then

$$\begin{cases} d(g\Pi_1 x_1, Sx_1) = \text{dist}(A, B), \\ d(g\Pi_1 x_2, Sx_2) = \text{dist}(A, B). \end{cases}$$

Since S is an ordered proximal contraction, there exists $\alpha \in [0, 1)$ such that

$$d(\Pi_1 x_1, \Pi_1 x_2) = d(g\Pi_1 x_1, g\Pi_1 x_2) \leq \alpha d(x_1, x_2).$$

Similarly, if $y_1, y_2 \in B_0$ with $y_1 \preceq y_2$, then

$$d(\Pi_2 y_1, \Pi_2 y_2) \leq \alpha d(y_1, y_2).$$

♣ By the assumption (v) of Theorem 2.6, there exist $x_0, x_1 \in A_0$ and $y_0, y_1 \in B_0$ with $x_0 \preceq x_1$ and $y_0 \preceq y_1$ such that $d(gx_1, Sx_0) = \text{dist}(A, B) = d(gy_1, Ty_0)$. Besides, by the definition of the mapping Π_1, we have $d(g\Pi_1 x_0, Sx_0) = \text{dist}(A, B)$. Because of the fact that S is an ordered proximal contraction, we

conclude that $x_1 = \Pi_1 x_0$ and so $x_0 \preceq \Pi_1 x_0$. Similarly, we obtain $y_0 \preceq \Pi_2 y_0$.
♣ Now define the mapping $\Pi : A_0 \cup B_0 \to A_0 \cup B_0$ with

$$\Pi z = \begin{cases} \Pi_1 z & \text{if } z \in A_0, \\ \Pi_2 z & \text{if } z \in B_0. \end{cases}$$

Then $\Pi(A_0) \subseteq A_0$ and $\Pi(B_0) \subseteq B_0$, that is, Π is *noncyclic* on $A_0 \cup B_0$. Let $(x, y) \in A_0 \times B_0$ be such that $x \preceq y$. Then we have

$$\begin{cases} d(g\Pi x, Sx) = \text{dist}(A, B), \\ d(g\Pi y, Ty) = \text{dist}(A, B). \end{cases}$$

Since the pair $(S; T)$ forms an ordered proximal cyclic contraction, we obtain

$$d(\Pi x, \Pi y) = d(g\Pi_1 x, g\Pi_2 y) \leq \beta d(x, y) + (1 - \beta)\text{dist}(A, B).$$

♣ For the considered elements $(x_0, y_0), (x_1, y_1) \in A_0 \times B_0$ which satisfy the condition (v) since $x_0 \preceq \Pi_1 x_0$ and Π_1 is monotone nondecreasing, the sequence $\{\Pi_1^n x_0\}$ is increasing. Similarly, the sequence $\{\Pi_2^n y_0\}$ is also increasing. It now follows from the proof of Theorem 1.1 that the sequences $\{\Pi_1^n x_0\}$ and $\{\Pi_2^n y_0\}$ are Cauchy. Let $(x^\star, y^\star) \in A \times B$ be such that

$$\Pi_1^n x_0 \to x^\star, \quad \Pi_2^n y_0 \to y^\star.$$

If we prove that $(x^\star, y^\star) \in A_0 \times B_0$ then by a similar argument of the proof of Theorem 1.1 we deduce that $x^\star$ and $y^\star$ are the fixed points of Π_1 and Π_2, respectively. To show this, we note that since $x_0 \preceq y_0$ we have

$$d(\Pi x_0, \Pi y_0) \leq \beta d(x_0, y_0) + (1 - \beta)\text{dist}(A, B).$$

Since

$$\begin{cases} d(g\Pi x_0, Sx_0) = \text{dist}(A, B), \\ d(g\Pi y_0, Ty_0) = \text{dist}(A, B), \end{cases}$$

and the pair $(S; T)$ forms a proximally increasing, we conclude that $g\Pi x_0 \preceq g\Pi y_0$. By the fact that g^{-1} is increasing, $\Pi x_0 \preceq \Pi y_0$. Again, since the pair $(S; T)$ forms an ordered proximal cyclic contraction, we obtain

$$\begin{aligned} d(\Pi^2 x_0, \Pi^2 y_0) &\leq \beta d(\Pi x_0, \Pi y_0) + (1 - \beta)\text{dist}(A, B) \\ &\leq \beta^2 d(x_0, y_0) + (1 - \beta^2)\text{dist}(A, B). \end{aligned}$$

Continuing this process and by induction, we conclude that

$$d(\Pi^n x_0, \Pi^n y_0) \leq \beta^n d(x_0, y_0) + (1 - \beta^n)\text{dist}(A, B).$$

Letting $n \to \infty$ in above inequality, we obtain $d(x^\star, y^\star) = \text{dist}(A, B)$, that is, $(x^\star, y^\star) \in A_0 \times B_0$. Hence,

$$\begin{aligned} d(gx^\star, Sx^\star) &= d(g\Pi_1 x^\star, Sx^\star) = \text{dist}(A, B), \\ d(gy^\star, Ty^\star) &= d(g\Pi_2 y^\star, Ty^\star) = \text{dist}(A, B), \\ d(x^\star, y^\star) &= \text{dist}(A, B). \end{aligned}$$

Finally, if for each $n \in \mathbb{N}$ we set $x_n = \Pi^n x_0$ and $y_n = \Pi^n y_0$, then

$$\begin{aligned} d(gx_{n+1}, Sx_n) &= \mathrm{dist}(A,B), \\ d(gy_{n+1}, Ty_n) &= \mathrm{dist}(A,B), \\ (x_n, y_n) &\to (x^\star, y^\star). \end{aligned}$$

□

Theorem 3.2. *Theorem 2.7 is a straightforward consequence of Theorem 1.2.*

Proof. Since the pair (A,B) has the P-property, it follows from Lemma 3.1 of [2] that both A_0 and B_0 are closed. Moreover, if $x \in A_0$, then there exists an element $v \in B_0$ such that $d(x,v) = \mathrm{dist}(A,B)$. We note that if there is another element $v' \in B_0$ for which $d(x,v') = \mathrm{dist}(A,B)$, then from the fact that (A,B) has the P-property, we must have $v = v'$. So, we can define a mapping $g : A_0 \to B_0$ such that

$$d(x, gx) = \mathrm{dist}(A,B), \quad \forall x \in A_0.$$

It is worth noticing that for any $u_1, u_2 \in A_0$, we have $d(u_1, gu_1) = \mathrm{dist}(A,B) = d(u_2, gu_2)$ which ensures that

$$d(u_1, u_2) = d(gu_1, gu_2), \quad \forall u_1, u_2 \in A_0,$$

that is, g is an isometry. Hence, g is a bijective isometry mapping. Now consider the self-mapping $g^{-1}T : A_0 \to A_0$. Here, we check the conditions of Theorem 1.1 for the self mapping $g^{-1}T : A_0 \to A_0$.

♠ Let $x, y \in A_0$ be such that $x \preceq y$. Since g^{-1} is an isometry, we conclude that

$$\psi\Big(d\big((g^{-1}T)x, (g^{-1}T)y\big)\Big) = \psi\big(d(Tx,Ty)\big) \leq \psi\big(d(x,y)\big) - \varphi\big(d(x,y)\big),$$

where $\varphi, \psi \in \Psi$.

♠ It follows from the assumption (ii) of Theorem 2.7 that there exist the elements $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = \mathrm{dist}(A,B)$. By the fact that $d(x_1, gx_1) = \mathrm{dist}(A,B)$ and that (A,B) has the P-property, we obtain $gx_1 = Tx_0$ and so, $x_1 = (g^{-1}T)x_0$ which implies that

$$x_0 \preceq (g^{-1}T)x_0.$$

♠ Let $x, y \in A_0$ be such that $x \preceq y$. Since $T(A_0) \subseteq B_0$ there are two points $u, v \in A_0$ such that

$$d(u, Tx) = \mathrm{dist}(A,B) = d(v, Ty).$$

Because T is proximally increasing, we must have $u \preceq v$. Besides, from the definition of the mapping g we have $gu = Tx$ and $gv = Ty$ and hence

$$(g^{-1}T)x = u \preceq v = (g^{-1}T)y,$$

which implies that the self mapping $g^{-1}T$ is monotone nondecreasing.
Thereby, all of the assumptions of Theorem 1.1 hold and the self mapping

$g^{-1}T : A_0 \to A_0$ has a fixed point, called $x^\star \in A_0$, that is, $g^{-1}Tx^\star = x^\star$ which ensures that $Tx^\star = gx^\star$. Hence,

$$d(x^\star, Tx^\star) = d(x^\star, gx^\star) = \text{dist}(A, B).$$

On the other hand if we define $x_n = \big(g^{-1}T\big)x_{n-1}$ for any $n \in \mathbb{N}$, then $x_n \to x^\star$. In this case we have $gx_n = Tx_{n-1}$ and so

$$d(x_n, Tx_{n-1}) = d(x_n, gx_n) = \text{dist}(A, B),$$

and the result follows. □

4. Concluding Remarks

It was proved by Sadiq Basha that in the setting of compete partially ordered metric spaces a pair of ordered proximal contractions which are proximally increasing has a common best proximity point (see Theorem 2.6). Moreover, an existence and convergence result of a best proximity point for proximally increasing nonself mappings was established by Pragadeeswarar and Maruda using a geometric concept of P-property (see Theorem 2.7).

We have proved that these existence results are straightforward consequences of Theorem 1.1 and Theorem 1.2, respectively.

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(Moosa Gabeleh) Department of Mathematics, Ayatollah Boroujerdi University, Boroujerd, Iran.
Email address: Gabeleh@abru.ac.ir

(Jack Markin) 1440 8th St. Golden, Co 80401, USA; jmarkin@newmexico.com
Email address: jmarkin@newmexico.com