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Research Paper

BIPOLAR MULTIPLICATIVE METRIC SPACES AND FIXED POINT THEOREMS OF COVARIANT AND CONTRAVARIANT MAPPINGS

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ABSTRACT. The definition of bipolar multiplicative metric space is introduced in this article, and in this space some properties are derived. Multiplicative contractions for covariant and contravariant maps are defined and fixed points are obtained. Also, some fixed point results of covariant and contravariant maps satisfying multiplicative contraction conditions are proved for bipolar multiplicative metric spaces. Moreover, Banach contraction principle and Kannan fixed point theorem are generalized.

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1. Introduction

A. E. Bashirov et al introduced the notion of multiplicative metric spaces in [3]. Topological properties of multiplicative metric spaces were derived and fixed point results of multiplicative contraction mappings were proved by M. Ozavsar, and A. C. Cevikel in [13]. There are many articles appeared for fixed point theory in multiplicative metric spaces, see [1, 2, 4, 5, 7].

Definition 1.1. [3] Let S be a non empty set. A multiplicative metric is a mapping $d : S \times S \to [1, \infty)$ satisfying the following axioms.

(i) $d(s,t) = 1$ if and only if $s = t$ in S,
(ii) $d(s,t) = d(t,s)$, $\forall\ s,t \in S$,
(iii) $d(s,t) \leq d(s,r)d(r,t)$, $\forall\ s,t,r \in S$.

The pair (S,d) is called a multiplicative metric space.

The notion of bipolar metric space has introduced by A. Mutlu and U. Gurdal [11], giving a new definition of distance measurement between the

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members of two separate sets. Bipolar metric space is a metric space generalization. Many articles are appearing for fixed point theory in bipolar metric spaces, see for example [8, 9, 12, 14] and the references therein.

Definition 1.2. [11] Let S and T be two non empty sets. A bipolar metric is a mapping $D : S \times T \to [0, \infty)$ satisfying the following axioms.

(I) $D(s,t) = 0 \Rightarrow s = t$, whenever $(s,t) \in S \times T$,
(II) $s = t \Rightarrow D(s,t) = 0$, whenever $(s,t) \in S \times T$,
(III) $D(s,t) = D(t,s)$, $\forall\ s,t \in S \cap T$,
(IV) $D(s_1,t_2) \leq D(s_1,t_1) + D(s_2,t_1) + D(s_2,t_2)$, $\forall\ s_1, s_2 \in S$, and $t_1, t_2 \in T$.

The triple (S, T, D) is called a bipolar metric space.

Proposition 1.3. [11] *Let (S, d) be a metric space. Then (S, d) is complete if and only if the corresponding bipolar metric space (S, S, d) is complete.*

In this paper, by extending the domain of multiplicative metric to a Cartesian product of two non-empty sets, we present a new definition of bipolar multiplicative metric space that generalizes the notion of multiplicative metric space. We derive some properties of bipolar multiplicative metric spaces. Also, we prove some fixed point results of covariant and contravariant maps satisfying various types of multiplicative contraction conditions in a bipolar multiplicative metric space. We shall also convert fixed point results from bipolar multiplicative metric spaces to bipolar metric spaces through exponential transformation. Moreover, we generalize the Banach contraction principle (see [10]), and Kannan fixed point result (see [6]).

2. **Bipolar multiplicative metric spaces**

Definition 2.1. Let S and T be two non empty sets. A bipolar multiplicative metric is a mapping $d : S \times T \to [1, \infty)$ satisfying the following conditions.

(I) $d(s,t) = 1 \Rightarrow s = t$, whenever $(s,t) \in S \times T$,
(II) $s = t \Rightarrow d(s,t) = 1$, whenever $(s,t) \in S \times T$,
(III) $d(s,t) = d(t,s)$, $\forall\ s,t \in S \cap T$,
(IV) $d(s_1,t_2) \leq d(s_1,t_1)d(s_2,t_1)d(s_2,t_2)$, $\forall\ s_1, s_2 \in S$, and $t_1, t_2 \in T$.

The triple (S, T, d) is called a bipolar multiplicative metric space(or, BMMS).

Remark 2.2. Let (S, T, d) be a BMMS. If $S \cap T = \emptyset$, then (S, T, d) is called disjoint. The space (S, T, d) is said to be a joint if $S \cap T \neq \emptyset$. The sets T and S are called right pole and left pole of (S, T, d), respectively.

Example 2.3. Let $S = (1, \infty)$, $T = (0, 1]$. Define $d : S \times T \to [0, \infty)$ as $d(s,t) = |\frac{s^2}{t^2}|_*$, whenever $(s,t) \in S \times T$, where $|.|_* : R^+ \to R^+$ is defined on a set of positive real numbers R^+ as follows: $|z| = z$ if $z \geq 1$ and $|z| = \frac{1}{z}$ if $z < 1$. Then (S, T, d) is a disjoint BMMS.

Remark 2.4. Let (S, d) be a multiplicative metric space, then (S, S, d) is a BMMS. Conversely, if (S, T, d) is a BMMS such that $S = T$, then (S, d) is a multiplicative metric space.

Definition 2.5. The opposite of a BMMS (S, T, d) is defined as the BMMS $(T, S, \overline{d})$, where the function $\overline{d} : T \times S \to [1, \infty)$ is defined as $\overline{d}(t, s) = d(s, t)$.

Definition 2.6. Let (S, T, d) be a BMMS. Where points of the sets T, S, and $S \cap T$ are called right, left, and central points respectively. A sequence that contains only right(or left, or central) points is called a right (or left, or central) sequence in (S, T, d).

Definition 2.7. Let (S, T, d) be a BMMS. A left sequence $(s_n)_{n=1}^{\infty}$ multiplicative converges to a right point t(or $(s_n)_{n=1}^{\infty} \to t$) if and only if for every $\epsilon > 1$, there exists an integer $n_0 \in N$ such that $d(s_n, t) < \epsilon$, $\forall\ n \geq n_0$. Also a right sequence $(t_n)_{n=1}^{\infty}$ multiplicative converges to a left point s (or $(t_n)_{n=1}^{\infty} \to s$) if and only if for every $\epsilon > 1$, there exists an integer $n_0 \in N$ such that $d(s, t_n) < \epsilon$, $\forall\ n \geq n_0$. When it is given $(k_n)_{n=1}^{\infty} \to l$ for a BMMS (S, T, d) without precise data about the sequence, this means that either $(k_n)_{n=1}^{\infty}$ is a right sequence and l is a left point, or $(k_n)_{n=1}^{\infty}$ is a left sequence and l is a right point.

Lemma 2.8. *Let (S, T, d) be a BMMS. Then a left sequence $(s_n)_{n=1}^{\infty}$ multiplicative converges to a right point t if and only if $d(s_n, t) \to 1$ in $(R^+, |.|_*)$, and also a right sequence $(t_n)_{n=1}^{\infty}$ multiplicative converges to a left point s if and only if $d(s, t_n) \to 1$ in $(R^+, |.|_*)$.*

Proof. Let $(s_n)_{n=1}^{\infty}$ be a left sequence, and $(s_n)_{n=1}^{\infty} \to t \in T$. For every $\epsilon > 1$, there exists an integer $n_0 \in N$ such that, for all $n \geq n_0$, $d(s_n, t) < \epsilon$. Hence

$$1 \leq d(s_n, t) < \epsilon,\ \forall\ n \geq n_0.$$

Since $|d(s_n, t)|_* < \epsilon$, $\forall\ n \geq n_0$, then $d(s_n, t) \to 1$ as $n \to \infty$ in $(R^+, |.|_*)$. The converse is also true. Obviously, a right sequence $(t_n)_{n=1}^{\infty}$ multiplicative converges to a left point s if and only if $d(s, t_n) \to 1$ in $(R^+, |.|_*)$ and this complete the proof. □

Lemma 2.9. *Let (S, T, d) be a BMMS. If a central point is a multiplicative limit of a sequence, then it is the unique multiplicative limit of the sequence.*

Proof. Let $(s_n)_{n=1}^{\infty}$ be a left sequence, $(s_n)_{n=1}^{\infty} \to s \in S \cap T$, and $(s_n)_{n=1}^{\infty} \to t \in T$. For every $\epsilon > 1$, there exists an integer $n_0 \in N$ such that, for all $n \geq n_0$, we have $d(s_n, s) < \sqrt{\epsilon}$, and $d(s_n, t) < \sqrt{\epsilon}$, and then

$$1 \leq d(s, t) \leq d(s, s) d(s_n, s) d(s_n, t) < 1.\sqrt{\epsilon}.\sqrt{\epsilon} = \epsilon.$$

Since $\epsilon > 1$ is arbitrary, we have $d(s, t) = 1$ which implies $s = t$. □

Lemma 2.10. *Let (S, T, d) be a BMMS. If a left sequence $(s_n)_{n=1}^{\infty}$ multiplicative converges to t and a right sequence $(t_n)_{n=1}^{\infty}$ multiplicative converges to s, then $d(s_n, t_n) \to d(s, t)$ as $n \to \infty$.*

Proof. Let $(s_n)_{n=1}^{\infty} \to t \in T$, and $(t_n)_{n=1}^{\infty} \to s \in S$. For every $\epsilon > 1$, there exists an integer $n_0 \in N$ such that, for all $n \geq n_0$, we have $d(s_n, t) < \sqrt{\epsilon}$, and $d(s, t_n) < \sqrt{\epsilon}$, then

$$d(s,t) \leq d(s,t_n)d(s_n,t_n)d(s_n,t)$$

implies

$$\frac{d(s,t)}{d(s_n,t_n)} \leq d(s,t_n)d(s_n,t),$$

and also

$$d(s_n,t_n) \leq d(s_n,t)d(s,t)d(s,t_n)$$

implies

$$\frac{d(s_n,t_n)}{d(s,t)} \leq d(s,t_n)d(s_n,t).$$

By the definition of $|.|_*$,

$$|\frac{d(s_n,t_n)}{d(s,t)}|_* \leq d(s,t_n)d(s_n,t) < \epsilon, \forall n \geq n_0,$$

and hence $d(s_n, t_n) \to d(s, t)$ as $n \to \infty$ in $(R^+, |.|_*)$. □

Definition 2.11. Let (S_1, T_1) and (S_2, T_2) be two bipolar multiplicative metric spaces(or,BMMSs), and $f : S_1 \cup T_1 \to S_2 \cup T_2$.

(i) If $f(S_1) \subseteq S_2$ and $f(T_1) \subseteq T_2$, then f is called a covariant map from (S_1, T_1) to (S_2, T_2), and we write $f : (S_1, T_1) \rightrightarrows (S_2, T_2)$.
(ii) If $f(S_1) \subseteq T_2$ and $f(T_1) \subseteq S_2$, then f is called a contravariant map from (S_1, T_1) to (S_2, T_2), and we write $f : (S_1, T_1) \rightleftarrows (S_2, T_2)$.

Remark 2.12. Suppose d_1, and d_2 be two bipolar multiplicative metrics on (S_1, T_1) and (S_2, T_2) respectively. We can also use the symbols $f : (S_1, T_1, d_1) \rightrightarrows (S_2, T_2, d_2)$ and $f : (S_1, T_1, d_1) \rightleftarrows (S_2, T_2, d_2)$ in the place of $f : (S_1, T_1) \rightrightarrows (S_2, T_2)$ and $f : (S_1, T_1) \rightleftarrows (S_2, T_2)$.

Definition 2.13. Let (S_1, T_1, d_1) and (S_2, T_2, d_2) be two BMMSs.

(i) A map $f : (S_1, T_1, d_1) \rightrightarrows (S_2, T_2, d_2)$ is called left multiplicative continuous at a point $s_0 \in S_1$, if for every $\epsilon > 1$, there exists $\delta > 1$ such that, the condition $d_1(s_0, t) < \delta$ implies that $d_2(f(s_0), f(t)) < \epsilon$, when $t \in T_1$.
(ii) A map $f : (S_1, T_1, d_1) \rightrightarrows (S_2, T_2, d_2)$ is called right multiplicative continuous at a point $t_0 \in T_1$, if for every $\epsilon > 1$, there exists $\delta > 1$ such that, the condition $d_1(s, t_0) < \delta$ implies that $d_2(f(s), f(t_0)) < \epsilon$, when $s \in S_1$.
(iii) A covariant map $f : (S_1, T_1, d_1) \rightrightarrows (S_2, T_2, d_2)$ is called multiplicative continuous if it is left multiplicative continuous at each point $s \in S_1$, and right multiplicative continuous at each point $t \in T_1$.

(iv) A contravariant map $f : (S_1, T_1, d_1) \rightleftarrows (S_2, T_2, d_2)$ is multiplicative continuous if and only if it is multiplicative continuous as a covariant map $f : (S_1, T_1, d_1) \rightrightarrows (T_2, S_2, \overline{d_2})$.

Remark 2.14. From the previous definition we have that a covariant or contravariant map f from (S_1, T_1, d_1) to (S_2, T_2, d_2) is multiplicative continuous if and only if a sequence $(k_n)_{n=1}^{\infty} \to l$ in (S_1, T_1, d_1) implies $(f(k_n))_{n=1}^{\infty} \to f(l)$ in (S_2, T_2, d_2)

Definition 2.15. Let (S, T, d) be a BMMS.

(i) A sequence (s_n, t_n) on the set $S \times T$ is called a bisequence on (S, T, d).
(ii) If both $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ multiplicative converges, then the bisequence (s_n, t_n) is called multiplicative convergent. If both $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ multiplicative converges to a same point $s \in S \cap T$, then the bisequence is called multiplicative biconvergent.
(iii) A bisequence (s_n, t_n) on (S, T, d) is called a multiplicative Cauchy bisequence, if for each $\epsilon > 1$, there is an $n_0 \in N$ such that $d(s_n, t_m) < \epsilon \ \forall \ n, m \geq n_0$.

Lemma 2.16. *Let (S, T, d) be a BMMS. Then (s_n, t_n) is a multiplicative Cauchy bisequence if and only if $d(s_n, t_m) \to 1$ as $n, m \to \infty$.*

Proof. Let (s_n, t_n) be a multiplicative Cauchy bisequence. For every $\epsilon > 1$, there exists an integer $n_0 \in N$ such that for all $n, m \geq n_0$ we have $1 \leq d(s_n, t_m) < \epsilon$ so that $|d(s_n, t_m)|_* < \epsilon$, for all $n, m \geq n_0$. Thus $d(s_n, t_m) \to 1$ as $n, m \to \infty$ in $(R^+, |.|_*)$. The converse is also true. □

Proposition 2.17. *Let (S, T, d) be a BMMS. Then every multiplicative biconvergent bisequence is a multiplicative Cauchy bisequence.*

Proof. Let (s_n, t_n) be a bisequence, which is multiplicative biconvergent to a point $s \in S \cap T$. For every $\epsilon > 1$, there exists an integer $n_0 \in N$ such that for every $n \geq n_0$, $d(s_n, s) < \sqrt{\epsilon}$, and for every $m \geq n_0$, $d(s, t_m) < \sqrt{\epsilon}$. Then we have

$$d(s_n, t_m) \leq d(s_n, s)d(s, s)d(s, t_m) < \sqrt{\epsilon}.1.\sqrt{\epsilon} = \epsilon, \forall n, m \geq n_0.$$

So (s_n, t_n) is a multiplicative Cauchy bisequence. □

Proposition 2.18. *Let (S, T, d) be a BMMS. Then every multiplicative convergent multiplicative Cauchy bisequence is multiplicative biconvergent.*

Proof. Let (s_n, t_n) be a multiplicative Cauchy bisequence such that $(s_n)_{n=1}^{\infty}$ multiplicative convergent to t in T and $(t_n)_{n=1}^{\infty}$ multiplicative convergent to s in S. For every $\epsilon > 1$, there exists an integer $n_0 \in N$ such that $d(s_n, t) < \sqrt[3]{\epsilon}$, $d(s, t_n) < \sqrt[3]{\epsilon}$, for all $n \geq n_0$, and $d(s_n, t_m) < \sqrt[3]{\epsilon}$, for all $n, m \geq n_0$. Then

$$1 \leq d(s, t) \leq d(s, t_m)d(s_n, t_m)d(s_n, t) < \sqrt[3]{\epsilon}\sqrt[3]{\epsilon}\sqrt[3]{\epsilon} = \epsilon, \forall n, m \geq n_0.$$

Therefore $d(s, t) = 1$ and so that $s = t$. Then (s_n, t_n) is multiplicative biconvergent. □

Definition 2.19. A BMMS (S,T,d) is called complete, if every multiplicative Cauchy bisequence is multiplicative convergent, or equivalently, multiplicative biconvergent.

Definition 2.20. Let (S_1,T_1,d_1) and (S_2,T_2,d_2) be two BMMSs. A covariant map $f:(S_1,T_1,d_1)\rightrightarrows(S_2,T_2,d_2)$ such that $d(f(s),f(t))\leq(d(s,t))^{\lambda}$, for all $s\in S_1, t\in T_1$, where $\lambda\in(0,1)$, or, a contravariant map $f:(S_1,T_1,d_1)\rightleftarrows(S_2,T_2,d_2)$ such that $d(f(t),f(s))\leq(d(s,t))^{\lambda}$, $\forall\ s\in S_1, t\in T_1$, for some $\lambda\in(0,1)$, is called multiplicative contraction.

3. main results

Theorem 3.1. *Let (S,T,d) be a complete BMMS. If a covariant map $f:(S,T,d)\rightrightarrows(S,T,d)$ satisfies $d(f(s),f(t))\leq(d(s,t))^{\lambda}$, whenever $(s,t)\in S\times T$ and $\lambda\in(0,1)$, then the function $f:S\cup T\to S\cup T$ has a unique fixed point(UFP).*

Proof. Let $s_0\in S$, $t_0\in T$ and $s_{n+1}=f(s_n)$ and $t_{n+1}=f(t_n)$, for all $n\in N$. Then (s_n,t_n) is a bisequence on (S,T,d). By using the contraction condition:

$$\begin{aligned} d(s_n,t_n) &= d(f(s_{n-1}),f(t_{n-1})) \\ &\leq (d(s_{n-1},t_{n-1}))^{\lambda}\leq ...\leq(d(s_0,t_0))^{\lambda^n} \end{aligned}$$

$$\begin{aligned} d(s_n,t_{n+1}) &= d(f(s_{n-1}),f(t_n)) \\ &\leq (d(s_{n-1},t_n))^{\lambda}\leq ...\leq(d(s_0,t_1))^{\lambda^n}. \end{aligned}$$

For every $n,q\in N$ and hence,

$$\begin{aligned} d(s_{n+q},t_n) &\leq d(s_{n+q},t_{n+1})d(s_n,t_{n+1})d(s_n,t_n) \\ &\leq d(s_{n+q},t_{n+1})(d(s_0,t_1))^{\lambda^n}(d(s_0,t_0))^{\lambda^n} \\ &= d(s_{n+q},t_{n+1})M^{\lambda^n},\ (M=d(s_0,t_1)d(s_0,t_0)) \\ &\leq d(s_{n+q},t_{n+2})d(s_{n+1},t_{n+2})d(s_{n+1},t_{n+1})M^{\lambda^n} \\ &\leq d(s_{n+q},t_{n+2})M^{(\lambda^{n+1}+\lambda^n)} \\ &\leq ... \\ &\leq d(s_{n+q},t_{n+q})M^{(\lambda^{n+q-1}+...+\lambda^{n+1}+\lambda^n)} \\ &\leq M^{(\lambda^{n+q}+...+\lambda^{n+1}+\lambda^n)} \\ &\leq M^{\lambda^n\sum\limits_{z=0}^{\infty}\lambda^z} \\ &= M^{\frac{\lambda^n}{1-\lambda}}=K_n, \end{aligned}$$

where $K_n=M^{\frac{\lambda^n}{1-\lambda}}$. Similarly $d(s_n,t_{n+q})\leq K_n$, for all $n,q\in N$.
Let $\epsilon>1$. Since $\lambda\in(0,1)$, there exists $n_0\in N$ such that $K_{n_0}=M^{\frac{\lambda^{n_0}}{1-\lambda}}<$

$\sqrt[3]{\epsilon}$. Therefore,

$$\begin{aligned} d(s_n, t_m) &\leq d(s_n, t_{n_0})d(s_{n_0}, t_{n_0})d(s_{n_0}, t_m) \\ &\leq {K_{n_0}}^3 < \epsilon, \forall n \geq n_0 \text{ and } \forall m \geq n_0. \end{aligned}$$

So (s_n, t_n) is a multiplicative Cauchy bisequence. Since (S, T, d) is complete, then (s_n, t_n) multiplicative converges, and multiplicative biconverges to a point $k \in S \cap T$. Also, $f(t_n) = t_{n+1} \to k \in S \cap T$ as $n \to \infty$. For all $n \in N$

$$\begin{aligned} d(k, f(k)) &\leq d(k, f(t_n))d(f(t_n), f(t_n))d(f(t_n), f(k)) \\ &\leq d(k, t_{n+1})(d(t_n, t_n))^{\lambda}(d(t_n, k))^{\lambda} \\ &\leq d(k, t_{n+1})(d(t_0, t_0))^{\lambda^{n+1}}(d(t_n, k))^{\lambda}. \end{aligned}$$

By taking the limit in the above inequality, we get $d(k, f(k)) \leq 1$. Hence, $d(k, f(k)) = 1$ so that $f(k) = k$. Therefore, k is a fixed point of f.
If l is another fixed point of f, then $f(l) = l$ implies $l \in S \cap T$, and

$$d(k, l) = d(f(k), f(l)) \leq (d(k, l))^{\lambda} \leq ... \leq (d(k, l))^{\lambda^n}, \text{ for every } n = 1, 2, 3, ...$$

Therefore $d(k, l) = 1$ so that $k = l$, and hence f has a unique fixed point and this completes the proof. □

Corollary 3.2. *Let (S, T, D) be a complete bipolar metric space. If a covariant map $f : (S, T, D) \rightrightarrows (S, T, D)$ satisfies $D(f(s), f(t)) \leq \lambda D(s, t)$, whenever $(s, t) \in S \times T$, where $\lambda \in (0, 1)$, then the function $f : S \cup T \to S \cup T$ has a UFP.*

Proof. Suppose $d = \exp D$. Then (S, T, d) is a complete BMMS. Also $d(f(s), f(t)) \leq (d(s, t))^{\lambda}$, where $(s, t) \in S \times T$ and $\lambda \in (0, 1)$. The proof now follows from Theorem 3.1. □

The above corollary is Theorem 5.1 of [11]. Also, this Corollary generalizes a Banach contraction principle (see [10]).

Theorem 3.3. *Let (S, T, d) be a complete BMMS. If a contravariant map $f : (S, T, d) \rightleftarrows (S, T, d)$ satisfies $d(f(t), f(s)) \leq (d(s, t))^{\lambda}$, whenever $(s, t) \in S \times T$, where $\lambda \in (0, 1)$, then the function $f : S \cup T \to S \cup T$ has a UFP.*

Proof. Let $s_0 \in S$, $t_0 = f(s_0) \in T$, and $s_1 = f(t_0)$, and let $t_n = f(s_n)$ and $s_{n+1} = f(t_n)$, for all $n \in N$. Then (s_n, t_n) is a bisequence on (S, T, d). Hence,

$$\begin{aligned} d(s_n, t_n) &= d(f(t_{n-1}), f(s_n)) \\ &\leq (d(s_n, t_{n-1}))^{\lambda} \\ &= (d(f(t_{n-1}), f(s_{n-1})))^{\lambda} \\ &\leq (d(s_{n-1}, t_{n-1}))^{\lambda^2} \leq ... \leq (d(s_0, t_0))^{\lambda^{2n}} = K_n^{1-\lambda} \leq K_n, \end{aligned}$$

$(K_n = d(s_0, t_0)^{\frac{\lambda^{2n}}{1-\lambda}})$. So

$$\begin{aligned} d(s_{n+1}, t_n) &= d(f(t_n), f(s_n)) \\ &\leq (d(s_n, t_n))^{\lambda} \leq (d(s_0, t_0))^{\lambda^{2n+1}}. \end{aligned}$$

For all $n, q \in N$, we have

$$\begin{aligned}
d(s_{n+q}, t_n) &\leq d(s_{n+q}, t_{n+1}) d(s_{n+1}, t_{n+1}) d(s_{n+1}, t_n) \\
&\leq d(s_{n+q}, t_{n+1}) (d(s_0, t_0))^{\lambda^{2n+2}+\lambda^{2n+1}} \\
&\leq d(s_{n+q}, t_{n+2}) d(s_{n+2}, t_{n+2}) d(s_{n+2}, t_{n+1}) (d(s_0, t_0))^{\lambda^{2n+2}+\lambda^{2n+1}} \\
&\leq d(s_{n+q}, t_{n+2}) (d(s_0, t_0))^{\lambda^{2n+4}+\lambda^{2n+3}+\lambda^{2n+2}+\lambda^{2n+1}} \\
&\leq \dots \\
&\leq d(s_{n+q}, t_{n+q-1}) (d(s_0, t_0))^{\lambda^{2n+2q-2}+\dots+\lambda^{2n+1}} \\
&\leq (d(s_0, t_0))^{\lambda^{2n+2q-1}+\lambda^{2n+2q-2}+\dots+\lambda^{2n+1}} \\
&\leq (d(s_0, t_0))^{\lambda^{2n+1} \sum\limits_{z=0}^{\infty} \lambda^z} \\
&= (d(s_0, t_0))^{\frac{\lambda^{2n+1}}{1-\lambda}} \\
&= {K_n}^{\lambda} < K_n, \ (K_n = d(s_0, t_0)^{\frac{\lambda^{2n}}{1-\lambda}}).
\end{aligned}$$

Similarly

$$d(s_n, t_{n+q}) \leq K_n.$$

Let $\epsilon > 1$. From $0 < \lambda < 1$, there exists $n_0 \in N$ such that $K_{n_0} = d(s_0, t_0)^{\frac{\lambda^{2n_0+1}}{1-\lambda}} < \sqrt[3]{\epsilon}$. Then,

$$\begin{aligned}
d(s_n, t_m) &= d(s_n, t_{n_0}) d(s_{n_0}, t_{n_0}) d(s_{n_0}, t_m) \\
&\leq {K_{n_0}}^3 < \epsilon, \forall n \geq n_0 \text{ and } \forall m \geq n_0.
\end{aligned}$$

So (s_n, t_n) is a multiplicative Cauchy bisequence. Since (S, T, d) is complete, (s_n, t_n) multiplicative converges, and multiplicative biconverges to a point $k \in S \cap T$. Also, $f(s_n) = t_n \to k \in S \cap T$ as $n \to \infty$. Thus, for all $n \in N$,

$$\begin{aligned}
d(k, f(k)) &\leq d(k, f(s_n)) d(f(t_n), f(s_n)) d(f(t_n), f(k)) \\
&\leq d(k, t_n) (d(s_n, t_n))^{\lambda} (d(k, t_n))^{\lambda} \\
&\leq d(k, t_n) (d(s_0, t_0))^{\lambda^{2n+1}} (d(k, t_n))^{\lambda}, \forall n.
\end{aligned}$$

So $d(k, f(k)) \leq 1$, as $n \to \infty$ in the previous inequality. Therefore $d(k, f(k)) = 1$ so that $f(k) = k$. Hence k is a fixed point.
If l is another fixed point of f, then $f(l) = l$, $l \in S \cap T$, and

$$d(k, l) \ = \ d(f(k), f(l)) \leq (d(k, l))^{\lambda} \leq \dots \leq (d(k, l))^{\lambda^n}, \text{ for every } n = 1, 2, 3, \dots.$$

Therefore $d(k, l) = 1$ so that $k = l$. So f has a unique fixed point and this completes the proof. □

Theorem 3.4. *Let (S, T, d) be a complete BMMS. If a contravariant map $f : (S, T, d) \rightleftarrows (S, T, d)$ satisfies $d(f(t), f(s)) \leq [d(s, f(s)) d(f(t), t)]^{\lambda}$, whenever $(s, t) \in S \times T$, for some $\lambda \in (0, \frac{1}{2})$, then the function $f : S \cup T \to S \cup T$ has a UFP.*

Proof. Let $s_0 \in S$, $t_0 = f(s_0) \in T$, and $s_1 = f(t_0)$. Suppose, $t_n = f(s_n)$ and $s_{n+1} = f(t_n)$, for all $n \in N$. Then (s_n, t_n) is a bisequence on (S, T, d). For all $n \in N$, from

$$\begin{aligned} d(s_n, t_n) &= d(f(t_{n-1}), f(s_n)) \\ &\leq [d(s_n, f(s_n)) d(f(t_{n-1}), t_{n-1})]^{\lambda} \\ &= [d(s_n, t_n) d(s_n, t_{n-1})]^{\lambda} \end{aligned}$$

we conclude that

$$d(s_n, t_n) \leq [d(s_n, t_{n-1})]^{\frac{\lambda}{1-\lambda}},$$

and

$$\begin{aligned} d(s_n, t_{n-1}) &= d(f(t_{n-1}), f(s_{n-1})) \\ &\leq [d(s_{n-1}, f(s_{n-1})) d(f(t_{n-1}), t_{n-1})]^{\lambda} \\ &\leq [d(s_{n-1}, t_{n-1}) d(s_n, t_{n-1})]^{\lambda} \end{aligned}$$

so that

$$d(s_n, t_{n-1}) \leq [d(s_{n-1}, t_{n-1})]^{\frac{\lambda}{1-\lambda}}.$$

Therefore, by putting $\alpha = \frac{\lambda}{1-\lambda}$, we have

$$d(s_n, t_n) \leq (d(s_0, t_0))^{\alpha^{2n}}$$

and

$$d(s_n, t_{n-1}) \leq (d(s_0, t_0))^{\alpha^{2n-1}}.$$

For every $m, n \in N$,

$$\begin{aligned} d(s_n, t_m) &\leq d(s_n, t_n) d(s_{n+1}, t_n) d(s_{n+1}, t_m) \\ &\leq (d(s_0, t_0))^{\alpha^{2n} + \alpha^{2n+1}} d(s_{n+1}, t_m) \\ &\leq ... \\ &\leq (d(s_0, t_0))^{\alpha^{2n} + \alpha^{2n+1} + ... + \alpha^{2m-1}} d(s_m, t_m) \\ &\leq (d(s_0, t_0))^{\alpha^{2n} + \alpha^{2n+1} + ... + \alpha^{2m}}, \text{ if } m > n, \end{aligned}$$

and similarly, if $m < n$, then

$$d(s_n, t_m) \leq (d(s_0, t_0))^{\alpha^{2m+1} + \alpha^{2m+2} + ... + \alpha^{2n+1}}.$$

By $\alpha \in (0, 1)$, $d(s_n, t_m) \to 1$, as $n, m \to \infty$, we conclude that (s_n, t_n) is a multiplicative Cauchy bisequence. Since (S, T, d) is complete, (s_n, t_n) multiplicative converges, and multiplicative biconverges to a point $k \in S \cap T$. Hence, $f(s_n) = t_n \to k \in S \cap T$ as $n \to \infty$ implies $d(f(k), f(s_n)) \to d(f(k), k)$ as $n \to \infty$, by using Lemma 2.10. Also by taking the limit from

$$\begin{aligned} d(f(k), f(s_n)) &\leq [d(s_n, f(s_n)) d(f(k), k)]^{\lambda} \\ &= [d(s_n, t_n) d(f(k), k)]^{\lambda}, \end{aligned}$$

as $n \to \infty$, we get $d(f(k),k) \leq (d(f(k),k))^{\lambda}$. Since $0 < \lambda < \frac{1}{2}$, $d(f(k),k) = 1$, hence $f(k) = k$. Therefore k is a fixed point of f.
If l is another fixed point of f, then $f(l) = l$, $l \in S \cap T$, and hence,

$$d(k,l) = d(f(k),f(l)) \leq (d(k,fk)d(fl,l))^{\lambda} = (d(k,k)d(l,l))^{\lambda}$$

Therefore $d(k,l) = 1$ so that $k = l$. So f has a unique fixed point, and this completes the proof. □

Corollary 3.5. *Let (S,T,D) be a complete bipolar metric space. If a contravariant map $f : (S,T,D) \rightleftarrows (S,T,D)$ satisfies $D(f(t),f(s)) \leq \lambda[D(s,f(s)) + D(f(t),t)]$, whenever $(s,t) \in S \times T$, and $\lambda \in (0,\frac{1}{2})$, then the function $f : S \cup T \to S \cup T$ has a UFP.*

Proof. Suppose $d = \exp D$. Then (S,T,d) is a complete BMMS. Also $d(f(t),f(s)) \leq [d(s,f(s))d(f(t),t)]^{\lambda}$, whenever $(s,t) \in S \times T$, and $\lambda \in (0,\frac{1}{2})$. Now the proof follows from Theorem 3.5. □

The above corollary is Theorem 5.6 of [11]. Also, the above corollary generalized the Kannan fixed point theorem [6].

Example 3.6. Let $S = [1,\infty)$ and $T = \left[\frac{1}{4},1\right]$, and $d(s,t) = |\frac{s^2}{t^2}|_*$, where $(s,t) \in S \times T$. Then (S,T,d) is a complete BMMS. Define a covariant map $f : (S,T,d) \rightrightarrows (S,T,d)$ by $f(u) = u^{\frac{1}{4}}$, for all $u \in S \cup T$. Then,

$$\begin{aligned} d(f(s),f(t)) = d(s^{\frac{1}{4}},t^{\frac{1}{4}}) &= (|\frac{s^{\frac{1}{2}}}{t^{\frac{1}{2}}}|) \\ &= (|\frac{s^2}{t^2}|)^{\frac{1}{4}} \\ &= (d(s,t))^{\frac{1}{4}} \\ &\leq (d(s,t))^{\lambda}, \forall \lambda \in \left[\frac{1}{4},1\right), \ \forall\ (s,t) \in (S,T). \end{aligned}$$

By Theorem 3.1, f has a UFP 1.

4. **Conclusions**

Almost all fixed point results can be converted from bipolar multiplicative metric spaces to bipolar metric spaces through exponential transformation, as it has been illustrated in this article. Also, all fixed point theorems in bipolar metric spaces can be regarded as generalizations of fixed point theorems in metric spaces. Therefore, studies of fixed point outcomes in bipolar multiplicative metric spaces are significant.

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References

[1] A. A, N. Abdou, *Fixed point theorems for generalized contraction mappings in multiplicative metric spaces.* J. Nonlinear Sci. Appl. 9 (2016), 2347-2363.

[2] R. P. Agarwal, E. Karapinar, and B. Samet, *An essential remark on fixed point results on multiplicative metric spaces.* Fixed Point Theory Appl. 21 (2016).

[3] A.E. Bashirov, E.M. Kurpınar and A. Ozyapıcı, *Multiplicative calculus and its applications*, J. Math. Anal. Appl., 337(2008), 36-48.

[4] F. Gu, and Y. J. Cho, *Common fixed point results for four maps satisfying ϕ-contractive condition in multiplicative metric spaces*, Fixed Point Theory Appl., (2015), 19 pages.

[5] X. J. He, M. M. Song, and D. P. Chen, *Common fixed points for weak commutative mappings on a multiplicative metric space*, Fixed Point Theory Appl., 48(2014), 9 pages.

[6] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc., 60 (1968), 71-76.

[7] S. M. Kang, P. Kumar, S. Kumar, P. Nagpal, and S. K. Garg, *Common fixed points for compatible mappings and its variants in multiplicativemetric spaces.* Int. J. Pure Appl. Math., 102 (2015), 383-406.

[8] G. N. V. Kishore, K. P. R. Rao, H. Isik, B. S. Rao, and A. Sombabu, *Covariant mappings and coupled fixed point results in bipolar metric spaces*, Int. J. Nonlinear Anal. Appl.,12 (2021), 1-15.

[9] G. N. V. Kishore, R. P. Agarwal, B. S. Rao, and R. V. N. S. Rao, *Caristi type cyclic contraction and common fixed point theorems in bipolar metric spaces with applications*, Fixed Point Theory Appl., 21 (2018), 1-13.

[10] M. K. Khamsi, and W. A. Kirk, *An introduction to metric spaces and fixed point theory*, John Wiley, New York, 2001.

[11] A. Mutlu, and U. Gurdal, *Bipolar metric spaces and some fixed point theorems*, J. Nonlinear Sci. Appl., 9 (2016), 5362-5373.

[12] A. Mutlu, K. Ozkan, and U. Gurdal, *Coupled fixed point theorems on bipolar metric spaces*, Eur. J. Pure Appl. Math., 10 (2017), 655–667.

[13] M. Ozavsar, and A. C. Cevikel, *Fixed points of multiplicative contraction mappings on multiplicative metric spaces*, arXiv:1205.5131v1 [math.GM], 2012, 1-14.

[14] B. Srinuvasa Rao, and G. N. V. Kishore, *Common fixed point theorems in bipolar metric spaces with applications to integral equations*, Int. J. Eng. Technol., 7 (2018), 1022-1026.

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