Mathematical Analysis
& Convex Optimization

Vol. 3 (2022), No. 1, 105–107
https:\\ maco.lu.ac.ir
DOI: 10.52547/maco.3.1.11

Research Paper

A NOTE ON THE BEST APPROXIMATION IN SPACES OF AFFINE FUNCTIONS

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Abstract. The proximinality of certain subspaces of spaces of bounded affine functions is proved. The results presented here are some linear versions of an old result due to Mazur. For the proofs we use some sandwich theorems of Fenchel's duality theory.

MSC(2010): 41A65; 52A07.
Keywords: Best approximation, convex set, affine function, sandwich theorem.

1. Introduction

Let S be a metric space and T be a subspace of S. Then T is called proximinal in S if every element of S has a best approximation by elements of T i.e. for every $s \in S$ there exists $t_s \in T$ such that

$$d(s, t_s) = d(s, T) := \inf_{t \in T} d(s, t).$$

The proximinality problem for linear subspaces of normed (function) spaces [7] has been considered by many authors. One of the early results in this direction is due to Mazur:

Theorem 1.1. *Let X, Y be compact Hausdorff spaces and $\phi : X \to Y$ be a surjective continuous map. Let $\mathbf{C}(X), \mathbf{C}(Y)$ denote the Banach spaces of real valued continuous functions on X, Y with supremum norm. Then the image of $\mathbf{C}(Y)$ in $\mathbf{C}(X)$ under the canonical linear map induced by ϕ, is proximinal.*

Mazur's proof can be found in the Monograph of Semadeni [6] page 124. It uses the Hahn-Tong Sandwich Theorem [6, Theorem 6.4.4] for existence of continuous functions between upper and lower semicontinuous real valued functions. The Mazur result have been extended for spaces of complex valued functions by Pełczyński [5] and for vector valued functions by Olech [4] and Blatter [1]. There is also a generalization for vector spaces of 'continuous functions' on 'noncommutative spaces' in terms of C*-algebras [8]. For more details see [7, page 15].

In this short note we show that some analogs of the Mazur result are satisfied for spaces of bounded affine functions. Our proofs are linear versions of the Mazur proof. But instead of the Hahn-Tong Theorem we use some sandwich theorems of Fenchel's duality theory for existence of affine functions between concave and convex functions.

Date: Received: February 23, 2022, Accepted: June 20, 2022.

2. Main Results

For a nonempty convex set C we denote by $\mathcal{A}_{\mathrm{b}}(C)$ the normed space of all bounded affine real valued functions on C with supremum norm. If C is a compact convex subset of a topological vector space we denote by $\mathcal{A}_{\mathrm{c}}(C)$ the closed subspace of $\mathcal{A}_{\mathrm{b}}(C)$ containing all continuous affine functions. Let C, D be two (compact) convex sets and $\phi : C \to D$ be a (continuous) surjective affine map. Then ϕ induces an isometric linear isomorphism $\tilde{\phi}$ from $(\mathcal{A}_{\mathrm{c}}(D))$ $\mathcal{A}_{\mathrm{b}}(D)$ into $(\mathcal{A}_{\mathrm{c}}(C))$ $\mathcal{A}_{\mathrm{b}}(C)$ defined by $\tilde{\phi}(h) := h \circ \phi$. In what follows we identify $\mathcal{A}_{\mathrm{b}}(D)$ (resp. $\mathcal{A}_{\mathrm{c}}(D)$) as a closed subspace of $\mathcal{A}_{\mathrm{b}}(C)$ (resp. $\mathcal{A}_{\mathrm{c}}(C)$).

Theorem 2.1. *Let C, D be two convex sets and $\phi : C \to D$ be a surjective affine map. Then $\mathcal{A}_{\mathrm{b}}(D)$, as a subspace of $\mathcal{A}_{\mathrm{b}}(C)$ via $\tilde{\phi}$, is proximinal.*

Proof. Suppose that $f \in \mathcal{A}_{\mathrm{b}}(C)$. We must show that there exists $h_0 \in \mathcal{A}_{\mathrm{b}}(D)$ such that $\|f - h_0 \circ \phi\| = d$ where

$$d := \inf\{\|f - h \circ \phi\| : h \in \mathcal{A}_{\mathrm{b}}(D)\}.$$

Let $c := \sup_{y \in D} r(y)$ where

$$r(y) := \sup_{x, x' \in \phi^{-1}(y)} (f(x) - f(x')).$$

For $h \in \mathcal{A}_{\mathrm{b}}(D)$ and $x, x' \in \phi^{-1}(y)$ we have,

$$|f(x) - f(x')| \leq |f(x) - h \circ \phi(x)| + |f(x') - h \circ \phi(x')|.$$

This shows that

$$\|f - h \circ \phi\| \geq 2^{-1}c \tag{2.1}$$

and

$$d \geq 2^{-1}c. \tag{2.2}$$

Let $f^{\downarrow}, f^{\uparrow}$ be bounded real valued functions on D defined by

$$f^{\downarrow}(y) := \inf_{x \in \phi^{-1}(y)} f(x)$$

and

$$f^{\uparrow}(y) := \sup_{x \in \phi^{-1}(y)} f(x).$$

Then it can be checked that

$$r(y) = f^{\uparrow}(y) - f^{\downarrow}(y) \tag{2.3}$$

and

$$f^{\uparrow} - 2^{-1}c \leq f^{\downarrow} + 2^{-1}c. \tag{2.4}$$

We show that $f^{\downarrow}$ is a convex function: Let $t \in [0,1]$ and $y, y' \in D$. For every $\epsilon > 0$, there exist $x, x' \in C$ such that

$$\phi(x) = y, \qquad \phi(x') = y',$$

and

$$f^{\downarrow}(y) \leq f(x) < f^{\downarrow}(y) + \epsilon, \qquad f^{\downarrow}(y') \leq f(x') < f^{\downarrow}(y') + \epsilon.$$

We have

$$\phi(tx + (1-t)x') = ty + (1-t)y'.$$

Thus,

$$\begin{aligned} f^{\downarrow}(ty+(1-t)y') &\le f(tx+(1-t)x') \\ &= tf(x)+(1-t)f(x') \\ &\le [tf^{\downarrow}(y)+t\epsilon]+[(1-t)f^{\downarrow}(y')+(1-t)\epsilon] \\ &= tf^{\downarrow}(y)+(1-t)f^{\downarrow}(y')+\epsilon. \end{aligned}$$

Since ϵ was arbitrary, we have

$$f^{\downarrow}(ty+(1-t)y') \le tf^{\downarrow}(y)+(1-t)f^{\downarrow}(y').$$

Similarly, it is proved that $f^{\uparrow}$ is concave. Now, by the Sandwich theorem [2, Corollary 2.4.1] there is an affine function $h_0 : D \to \mathbb{R}$ with

$$f^{\uparrow} - 2^{-1}c \le h_0 \le f^{\downarrow} + 2^{-1}c. \tag{2.5}$$

Thus $h_0 \in \mathcal{A}_{\mathrm{b}}(D)$ and $\|f - h_0 \circ \phi\| \le 2^{-1}c$. Hence, by (2.1) and (2.2), $\|f - h_0 \circ \phi\| = d$. □

A continuous version of Theorem 2.1 is as follows.

Theorem 2.2. *Let C be an arbitrary compact convex set and D be a compact convex subset of a Fréchet topological vector space. Let $\phi : C \to D$ be a surjective continuous affine map. Then $\mathcal{A}_{\mathrm{c}}(D)$, as a subspace of $\mathcal{A}_{\mathrm{c}}(C)$ via $\tilde{\phi}$, is proximinal.*

Proof. Suppose that $f \in \mathcal{A}_{\mathrm{c}}(C)$. Let

$$d := \inf\{\|f - h \circ \phi\| : h \in \mathcal{A}_{\mathrm{c}}(D)\}.$$

Also, let $r(y), c, f^{\uparrow}, f^{\downarrow}$ be as in the proof of Theorem 2.1. Thus, for every $h \in \mathcal{A}_{\mathrm{c}}(D)$, (2.1), (2.2), (2.3), and (2.4) are satisfied. It follows from [6, Lemma 7.5.5] that $f^{\downarrow}$ and $f^{\uparrow}$ are respectively lower and upper semicontinuous functions. By the Sandwich theorem [3, Theorem 6(2)] of Noll, there exists a continuous affine function $h_0 : D \to \mathbb{R}$ satisfying (2.5). Thus $h_0 \in \mathcal{A}_{\mathrm{c}}(D)$ and $\|f - h_0 \circ \phi\| = d$. The proof is complete. □

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