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Research Paper

# CONVOLUTION WEIGHTED ORLICZ ALGEBRAS IN CONTEXT OF $\sigma$-COMPACT GROUPS

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ABSTRACT. In this paper, some sufficient condition for a weighted Orlicz space, $L^{\Phi}_{w}(G)$, to be a Banach algebra with convolution as multiplication in context of a $\sigma$-compact groups. We also for a class of Orlicz spaces, obtain an equivalent condition, such that a weighted Orlicz space to be a convolution Banach algebra. This resultes generalized some known results in Lebesgue spaces.

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**Keywords:** locally compact group, $\sigma$-compact group, convolution algebra, weighted Orlicz space, Orlicz spaces.

## 1. Introduction and Background

The condition that the Lebesgue spaces become an algebra by convolution multiplication has always been considered. It is known that for each locally compact group $G$, $L^1(G)$ is a convolution Banach algebra, and for each $p > 1$, $L^p(G)$ is a convolution algebra if and only if $G$ is compact ([4], [15]). Kuznetsova in [8] proved that $L^1_w(G)$ is convolution Banach algebra if and only if $w$ is submultiplicative, and in the case of $G$ is abelian for a $p > 1$ and a weight $w$ $L^p_w(G)$ is a convolution algebra if and only if $G$ is $\sigma$-compact.
Given that Orlicz spaces are a very important generalization of Lebesgue spaces, the natural question is when Orlicz spaces or weighted Orlicz spaces are convolution Banach algebra. For a locally compact abelian group $G$ and a $\Delta_2$-regular Young function $\Phi$, H. Hudzik, A. Kamiska and J. Musielak in [6] prove that the Orlicz space $L^{\Phi}(G)$ is a convolution Banach algebra if and only if $L^{\Phi}(G) \subseteq L^1(G)$, and this holds if and only if $G$ is compact or $\lim_{x\to 0^+} \frac{\Phi(x)}{x} > 0$. In [1, 13, 14] another necessary and sufficient condition for an Orlicz space $L^{\Phi}(G)$, and its weighted version $L^{\Phi}_w(G)$, to be a convolution Banach algebra is given. Also to observe similar results with the context of hypergroups see [2, 12].

In this paper, $G$ is a locally compact group and $\lambda$ is the left Haar measure on $G$. Let recall some properties of Orlicz spaces. For more details see three books [7, 10, 11] which are basic references for this subject.

A *Young function* is a convex even mapping $\Phi : \mathbb{R} \to [0, \infty]$ such that $\Phi(0) = \lim_{x\to 0} \Phi(x) = 0$ and $\lim_{x\to\infty} \Phi(x) = \infty$. A Young function $\Phi : \mathbb{R} \to [0, \infty)$ is called an *N-function* if it is continuous, $\Phi(x) = 0$ implies $x = 0$, $\lim_{x\to 0} \frac{\Phi(x)}{x} = 0$ and $\lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty$.

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The *complementary* of a Young function $\Phi$ is the function $\Psi$ defined by

$$\Psi(x) := \sup\{y|x| - \Phi(y) : y \geq 0\}, \quad (x \in \mathbb{R}).$$

If $\Psi$ is the complementary of a Young function $\Phi$, then $(\Phi, \Psi)$ is called a *Young pair*. If we denote the set of all Borel measurable complex-valued functions on $G$ by $\mathcal{B}(G)$, then the Orlicz space $L^\Phi(G)$ define by

$$L^\Phi(G) = \{f \in \mathcal{B}(G) : \exists \alpha > 0, \int_G \Phi(\alpha|f(x)|)\, d\lambda(x) < \infty\}.$$

For each Young pair $(\Phi, \Psi)$ we denote the set of all $f \in \mathcal{B}(G)$ with $\int_G \Phi(|f(x)|)\, d\lambda(x) \leq 1$ by $B_\Phi$. Then we can define for each $f \in L^\Phi(G)$ the *Orlicz norm* $\|f\|_\Phi$ and *Luxemburg norm* $\|f\|_{[\Phi]}$ respectively by

$$\|f\|_\Phi := \sup\left\{\int_G |f(x)g(x)|\, d\lambda(x) : g \in B_\Psi\right\}$$
$$\|f\|_{[\Phi]} := \inf\left\{\alpha > 0 : \frac{1}{\alpha} f \in B_\Phi\right\}.$$

If almost everywhere equal functions in $L^\Phi(G)$ consider the same, then $\|\cdot\|_\Phi$ and $\|\cdot\|_{[\Phi]}$ are equaivalent normes on $L^\Phi(G)$ with

$$\|f\|_{[\Phi]} \leq \|f\|_\Phi \leq 2\|f\|_{[\Phi]}. \tag{1.1}$$

For each $f \in L^\Phi(G)$ and $g \in L^\Psi(G)$, we have the *Hölder's inequality* for Orlicz spaces (see[10, Proposition 1, and the Remark after it, page 58]):

$$\int_G |f(x)g(x)|\, d\lambda(x) \leq 2\|f\|_{[\Phi]}\, \|g\|_{[\Psi]}. \tag{1.2}$$

A continuous positive function $w$ on $G$ called a *weight*. Simillar to other function spaces, the *weighted Orlicz space*, $L^\Phi_w(G)$, is defined by

$$L^\Phi_w(G) = \{f \in \mathcal{B}(G) : wf \in L^\Phi(G)\}.$$

$L^\Phi_w(G)$ with the norm $\|f\|_{\Phi,w} := \|wf\|_\Phi$ is a Banach space.

## 2. Main Results

Our motivation for giving the main result of this paper is to provide a generalization for [8, Theorem 1.1]. In that theorem it has been proved that a locally compact group $G$ is $\sigma$-compact if and only if for any $p > 1$ there exists a weight $w$ such that $w^{-q} * w^{-q} < w^{-q}$, if and only if for any $p > 1$ there exists a weight $w$ such that $L^\Phi_w(G)$ is a convolution Banach algebra. Recall that for any measurable functions $f$ and $g$ on $G$, the convolution product $f * g$ is defined by

$$(f * g)(x) := \int_G f(y)g(y^{-1}x)\, d\lambda(y)$$

for all $x \in G$, while this integral exists. Also the weighted Orlicz space $L^\Phi_w(G)$ is called a *convolution Banach algebra* if there exists a constant $D > 0$ such that

$$\|f * g\|_{\Phi,w} \leq D\, \|f\|_{\Phi,w}\, \|g\|_{\Phi,w},$$

for all $f, g \in L^\Phi_w(G)$.

In sequel, for each weight function $w$ on $G$, we define $\Omega_w : G \times G \to (0, \infty)$ by

$$\Omega_w(x, y) = \frac{w(yx)}{w(x)w(y)}.$$

Also, we say that a Young function $\Psi$ is satisfied the property $(\mathcal{A})$ whenever, for each $\sigma$-compact group $G$ there exists a weight $w$ on $G$ and there exists a real number $C > 0$ such that for all $v \in B_\Psi$, the function $H_v(y) := \|\Omega_w(\cdot, y)L_{y^{-1}}|v|\|_\Psi, (y \in G)$ belongs to $L^\Psi(G)$ and $\|H_v\|_\Psi \leq C$ where for each function $f$ on $G$ and each $y \in G$, the function $L_y f$ on $G$ define by $L_y f(x) = f(y^{-1}x)$.

**Remark 2.1.** By [8, Theorem 1.1] easily one can see that for each $1 < q < \infty$, the Young function $\Psi_q := |\cdot|^q$ satisfies the property $(\mathcal{A})$.
Indeed for each locally compact group $G$ by [8, Theorem 1.1] always there exists a weight $w$ such that $w^{-q} * w^{-q} < w^{-q}$. So for each $v \in B_{\Psi_q}$ we have

$$\begin{aligned}
\|H_v\|_{\Psi_q} &= \left(\int_G (H_v(y))^q\, dy\right)^{\frac{1}{q}} \\
&= \left(\int_G (\|\Omega_w(\cdot, y)L_{y^{-1}}|v|\|_{\Psi_q})^q\, dy\right)^{\frac{1}{q}} \\
&= \left(\int_G \int_G |\Omega_w(x, y)L_{y^{-1}}v(x)|^q\, dx\, dy\right)^{\frac{1}{q}} \\
&= \left(\int_G \int_G \frac{w(yx)^q}{w(x)^q w(y)^q} |v(yx)|^q\, dx\, dy\right)^{\frac{1}{q}} \\
&= \left(\int_G w(x)^q |v(x)|^q (\int_G \frac{1}{w(y)^q w(y^{-1}x)^q}\, dy)\, dx\right)^{\frac{1}{q}} \\
&= \left(\int_G w(x)^q |v(x)|^q (w^{-q} * w^{-q})(x)\, dx\right)^{\frac{1}{q}} \\
&\leq \left(\int_G |v(x)|^q x\right)^{\frac{1}{q}} \\
&= \|v\|_{\Psi_q} \leq 1.
\end{aligned}$$

Therefore $\Psi_q$ satisfies the property $(\mathcal{A})$ with $C = 1$.

Before stating the main thesis of this article we also recall that for a locally compact abelian group $G$, a weight function $w$ and a complementary pair of Young functions$(\Phi, \Psi)$ a function $\xi : G \to \mathbb{C} \setminus \{0\}$ is called a *generalized character* if $\xi(xy) = \xi(x)\,\xi(y)$ for all $x, y \in G$, and $\frac{\xi}{w} \in L^\Psi(G)$. The set of all generalized characters of $G$ is denoted by $\widehat{G_\Psi}(w)$.

**Theorem 2.2.** *Let $G$ be a locally compact group. Then, the following conditions are equivalent:*

(1) *$G$ is $\sigma$-compact.*
(2) *There are a complementary pair $(\Phi, \Psi)$ of $N$-functions in which for some weight function $w$, we have $\frac{1}{w} \in L^\Psi(G)$ and*

$$\Psi(\frac{1}{w}) * \Psi(\frac{1}{w}) \leq \Psi(\frac{1}{w}).$$

(3) *For all complementary pair* $(\Phi,\Psi)$ *of* $N$*-functions there exists some weight function* $w$ *such that* $\frac{1}{w}\in L^{\Psi}(G)$ *and*

$$\Psi(\frac{1}{w})*\Psi(\frac{1}{w})\leq\Psi(\frac{1}{w}).$$

*If* $G$ *is abelian, then the above conditions are equivalent to the following conditions:*

(4) *There are a complementary pair* $(\Phi,\Psi)$ *of* $N$*-functions in which* $\Psi$ *satisfies the property* $(\mathcal{A})$ *for some weight* $w$ *on* $G$*, and* $L^{\Phi}_w(G)$ *is a convolution Banach algebra.*

(5) *For each complementary pair* $(\Phi,\Psi)$ *of* $N$*-functions in which* $\Psi$ *satisfies the property* $(\mathcal{A})$ *for some weight* $w$ *on* $G$*,* $L^{\Phi}_w(G)$ *is a convolution Banach algebra.*

*Proof.* (2) $\Rightarrow$ (1): Let $(\Phi,\Psi)$ be a complementary pair of $N$-functions and $w$ be a weight on $G$ such that $\frac{1}{w}\in L^{\Psi}(G)$. Then for some $\alpha>0$, we have $\Psi(\frac{\alpha}{w})\in L^1(G)$. Since $\Psi(t)=0$ implies that $t=0$, and $\Psi(\frac{\alpha}{w})>0$ on $G$, we have

$$G=\{x\in G:\ \Psi(\frac{\alpha}{w})>0\},$$

and so thanks to [3, Proposition 2.20] and [4, Theorem 1.40], $G$ is $\sigma$-compact.
(1) $\Rightarrow$ (3): Let $G$ be a $\sigma$-compact group. By the proof of [8, Theorem 1.1], there is an integrable function $u>0$ such that $u*u\leq u$. Now for each $N$-function $\Psi$ if we put $w:=\dfrac{1}{\Psi^{-1}(u)}$, then we have $\frac{1}{w}\in L^{\Psi}(G)$ and $\Psi(\frac{1}{w})*\Psi(\frac{1}{w})\leq\Psi(\frac{1}{w})$.
Since (3) $\Rightarrow$ (2) is trivial, the conditions $1-3$ are equivalent.
(1) $\Rightarrow$ (5): Let $G$ be a $\sigma$-compact group. Then for all pairs $(\Phi,\Psi)$ of $N$-functions such that $\Psi$ satisfies the prperty $(\mathcal{A})$, there exists a weight $w$ on $G$ that satisfies the property $(\mathcal{A})$. Suppose that $f,g\in L^{\Phi}_w(G)$. Then for each $v\in B_{\Psi}$, since for almost every elements $y\in G$ we have $\|\Omega_w(\cdot,y)L_{y^{-1}}|v|\|_{\Psi}<\infty$, thanks to Hölder's inequality (1.2) and inequality (1.1) we have

$$\begin{aligned}\int_G |(f*g)(x)|\ w(x)\,|v(x)|\,d\lambda(x)&\\ &\leq\int_G|f(y)|\,w(y)\int_G|g(x)|\,w(x)\,\frac{w(yx)}{w(x)w(y)}\ |v(yx)|\,d\lambda(x)d\lambda(y)\\ &\leq 2\int_G|f(y)|\,w(y)\,\|fw\|_{\Phi}\|\Omega_w(\cdot,y)L_{y^{-1}}|v|\|_{\Psi}\,d\lambda(x)d\lambda(y)\\ &\leq 4\,\|f\|_{\Phi,w}\ \|g\|_{\Phi,w}\ \|H\|_{\Psi}\\ &\leq 4C\,\|g\|_{\Phi,w}\,\|f\|_{\Phi,w}.\end{aligned}$$

So we have $\|f*g\|_{\Phi,w}\leq 4C\,\|f\|_{\Phi,w}\,\|g\|_{\Phi,w}$ and $L^{\Phi}_w(G)$ is a convolution algebra.

Now, let $G$ be abelian.
(4) $\Rightarrow$ (1): Let $L^{\Phi}_w(G)$ be a convolution algebra. Since $G$ is abelian, by [9, Proposition. 5.7] $L^{\Phi}_w(G)$ is not radical. This implies that there exists a non-zero homomorphism $T$ from $L^{\Phi}_w(G)$ into $\mathbb{C}$, and so by [9, Theorem 5.2], there exists an element $\eta\in\widehat{G_{\Psi}}(w)$ such that

$$T(f)=\int_G f\,\eta\,d\lambda,\qquad (f\in L^{\Phi}_w(G)).$$

Hence we have $\frac{\eta}{w}\in L^{\Psi}(G)$. In other words, there exists an $\alpha>0$ such that $\Psi(\frac{|\eta|}{\alpha w})\in L^1(G)$. Therefore $G$ is $\sigma$-compact. $\square$

By [5, Proposition 5] and [9, Lamma 4.3], for any $N$-function $\Psi$, $L^\Psi(G)$ is a Banach algebra with respect to pointwise multiplication if and only if $L^\Psi(G) \subseteq L^\infty(G)$ if and only if $G$ is discrete. Using these facts, we can express the following conclusion.

**Corollary 2.3.** *Let $G$ be a discrete group. Then for any weight $w$ on $G$ such that*

(1) *for each $y \in G$, $\Omega_w(\cdot, y) \in B_\Psi$ and,*
(2) *if $H(y) = \|\Omega_w(\cdot, y)\|_\Psi, (y \in G)$, then $H \in L^\Psi(G)$,*

*$\ell^\Phi_w(G)$ is a convolution Banach algebras.*

*Proof.* Since $G$ is discrete, by [5, Proposition 5] and [9, Lamma 4.3], $\ell^\Psi(G)$ is Banach algebra with respect to pointwise multiplication. So there exsists a real number $B$ such that for each $f, g \in \ell^\Psi(G)$ we have $\|fg\|_\Psi \leq B\,\|f\|_\Psi \|fg\|_\Psi$. By substituting $\frac{1}{B}\Psi$ for $\Psi$, conditions (1) and (2) imply that if $v \in B_\Psi$, then

(1) for each $y \in G$,
$$\|\Omega_w(\cdot, y)L_{y^{-1}}|v|\|_\Psi \leq \|\Omega_w(\cdot, y)\|_\Psi \|L_{y^{-1}}|v|\|_\Psi \leq \|\Omega_w(\cdot, y)\|_\Psi, \text{ and}$$
(2) the function $H(y) = \|\Omega_w(\cdot, y)L_{y^{-1}}|v|\|_\Psi (y \in G)$, belongs to $B_\Psi$,

since the mapping $x \mapsto v(yx)$ belongs to $B_\Psi$. So the Young function $\Psi$ is satisfied the property $(\mathcal{A})$ and by Theorem 2.2, $\ell^\Phi_w(G)$ is a convolution Banach algebra. □

Before stating the next corollary, we remind that a Young function $\Phi$ satisfies $\Delta_2$-*condition* (and write $\Phi \in \Delta_2$) if there are $c > 0$ and $x_0 \geq 0$ such that
$$\Phi(2x) \leq c\,\Phi(x), \quad (x \geq x_0).$$

**Corollary 2.4.** *If $\Psi$ is globally $\Delta_2$–regular and $w$ is a weight on $\mathbb{Z}$ such that*
$$\exists \beta > 0, \forall m, n \in \mathbb{Z}, \forall v \in B_\Psi, \Omega_\omega(m, n)|v(m-n)| \leq \frac{\beta}{(|m|+1)(|n|+1)} \tag{2.1}$$
*then $\ell^\Phi_w(\mathbb{Z})$ is a convolution Banach algebras.*

*Proof.* Since $\Psi \in \Delta_2$ globally by [10, Chapter II, Corollary 5] there exist a real numbers $M > 0$ and $\alpha > 1$ such that for all $x \in \mathbb{R}$, $\Psi(x) \leq M|x|^\alpha$. So by (2.1) for each $n \in \mathbb{N}$ we have
$$\begin{aligned}
\int_{\mathbb{Z}} \Psi(\Omega_\omega(\cdot, n)L_{-n}|v|) &= \sum_{m=-\infty}^{+\infty} \Psi(\Omega_\omega(m, n)L_{-n}|v|) \\
&\leq \sum_{m=-\infty}^{+\infty} M\big|\Omega_\omega(m, n)L_{-n}|v|\big|^\alpha \\
&\leq \sum_{m=-\infty}^{+\infty} M\left(\frac{\beta}{(|m|+1)(|n|+1)}\right)^\alpha \\
&= \frac{M\beta^\alpha}{(1+|n|)^\alpha} \sum_{m=-\infty}^{+\infty} \frac{1}{(1+|m|)^\alpha} \\
&\leq M\beta^\alpha \sum_{m=-\infty}^{+\infty} \frac{1}{(1+|m|)^\alpha} \\
&= M\beta^\alpha(2\zeta(\alpha) - 1) < \infty,
\end{aligned}$$

where $\zeta$ is the Riemann zeta function.
On the other hand if $f \in B_\Phi$ we have $\sum_{m=-\infty}^{+\infty} |\Phi(f(m))| \leq 1$. So that for each $m \in \mathbb{Z}$ we have $|\Phi(f(m))| \leq 1$. Therefore for each $n \in \mathbb{Z}$ we see that

$$\begin{aligned}\|\Omega_\omega(\cdot, n)L_{-n}|v|\|_\Psi =& \sup\{\sum_{m=-\infty}^{+\infty} |\Omega_\omega(m,n)v(m-n)f(m)| : f \in B_\Phi\} \\ \leq& \Phi^{-1}(1) \sum_{m=-\infty}^{+\infty} |\Omega_\omega(m,n)v(m-n)|\end{aligned}$$

So for the function $H : y \mapsto \|\Omega_\omega(\cdot, n)L_{-n}|v|\|_\Psi$ we have

$$\begin{aligned}\int_{\mathbb{Z}} \Psi(H) =& \sum_{n=-\infty}^{+\infty} \Psi(H)(n) \\ \leq& \sum_{n=-\infty}^{+\infty} \Psi\left(\Phi^{-1}(1) \sum_{m=-\infty}^{+\infty} |\Omega_\omega(m,n)v(m-n)|\right) \\ \leq& \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} M \left|\Phi^{-1}(1)\Omega_\omega(m,n)v(m-n)\right|^\alpha \\ \leq& \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} M\Phi^{-1}(1)^\alpha \left(\frac{\beta}{(|m|+1)(|n|+1)}\right)^\alpha \\ =& M\Phi^{-1}(1)^\alpha \beta^\alpha (2\zeta(\alpha)-1)^2.\end{aligned}$$

If $\gamma = \max\{M\beta^\alpha(2\zeta(\alpha)-1), M\Phi^{-1}(1)^\alpha\beta^\alpha(2\zeta(\alpha)-1)^2\}$ then considering $\frac{\Psi}{\gamma}$ instead of $\Psi$ it is observed that $\Psi$ is true in the condition of of Theorem 2.2, so that $\ell_w^\Phi(\mathbb{Z})$ is a convolution Banach algebras. □

## References


[1] A. Bagheri Salec and S. M. Tabatabaie, Some Necessary and Sufficient Conditions for Convolution Weighted Orlicz Algebras. *Bull. Iran. Math. Soc.*, https://doi.org/10.1007/s41980-021-00655-y, 2021.

[2] A. Bagheri Salec, V. Kumar and S. M. Tabatabaie, Convolution property of Orlicz spaces on hypergroups. *Proc. Amer. Math. Soc.*, **150**(4): 1685–1696, 2022.

[3] G.B. Folland, *Real Analysis; Modern Techniques and Their Applications 2nd Edition*, John Wiley and Sons, Inc., New York, 1999.

[4] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, Springer New York, NY, 1979.

[5] H. Hudzik, *Orlicz spaces of essentially bounded functions and Banach-Orlicz algebras* , Arch. Math., **44** (1985) 535–538.

[6] H. Hudzik, A. Kamiska and J. Musielak, *On some Banach algebras given by a modular*, in: Alfred Haar Memorial Conference, Budapest, Colloquia Mathematica Societatis J anos Bolyai (North Holland, Amsterdam), **49** (1987) 445–463.

[7] M. A. Krasnosel'skii and Ja. B. Rutickii, Convex Functions and Orlicz Spaces. *Noordhoff, Groningen*, 1961.

[8] Yu. N. Kuznetsova, *Invariant weighted algebras* $L_p^w(G)$, Math. Notes, **84** (4) (2008) 529–537.

[9] A. Osançlıol and S. Öztop, *Weighted Orlicz algebras on locally compact groups*, J. Aust. Math. Soc., **99** (2015) 399–414.

[10] M.M. Rao and Z.D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, New York, 1991.

[11] M.M. Rao and Z.D. Ren, *Applications of Orlicz Spaces*, Marcel Dekker, New York, 2002.

[12] S.M. Tabatabaie, A.R. Bagheri Salec, Convolution of two weighted Orlicz spaces on hypergroups. *Revista Colombiana de Matemáticas*, **8**(1):115–120, 1961.

[13] S.M. Tabatabaie, A.R. Bagheri Salec and M. Zare Sanjari, *A note on Orlicz algebras*, Operators and Matrices, **54** (2) (2020) 117–128.

[14] S.M. Tabatabaie, A.R. Bagheri Salec and M. Zare Sanjari, *Remarks on weighted Orlicz spaces on locally compact groups*, Math. Inequal. Appl., **23** (3) (2020) 1015–1025.

[15] W. Zelazko, On the algebras $L^p$ of locally compact groups. *Colloq. Math.*, **8**(1):115–120, 1961.

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