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Research Paper

CONSTRUCTION OF NONNEGATIVE MATRIX FOR SPECIAL SPECTRUM

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Abstract. The construction of a nonnegative matrix for a given set of eigenvalues is one of the objectives of this paper. The generalization of the cases discussed in the previous papers as well as finding a recursive solution for the Suleimanova spectrum are other points that are studied in this paper.

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1. Introduction

A matrix A is called nonnegative if all its entries are nonnegative. The nonnegative inverse eigenvalue problem (NIEP) asks for necessary and sufficient conditions on a list $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of complex numbers in order that it be the spectrum of a nonnegative matrix. In this case, one says that σ is realizable and a nonnegative matrix A with spectrum σ is said to realize σ and it is referred to as a realizing matrix. There is a right and a left eigenvector associated with the Perron eigenvalue with nonnegative entries. The spectral radius of the nonnegative matrix A is denoted by $\rho(A)$. In addition s_k the k-th power sum of the eigenvalues λ_i and in the list $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, λ_1 is the Perron element. Some necessary conditions on the list of complex numbers $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ to be the spectrum of a nonnegative matrix are listed below.
(1) The Perron eigenvalue $\max\{|\lambda_i|; \lambda_i \in \sigma\}$ belongs to σ (Perron -Frobenius Theorem).
(2) The list σ is closed under complex conjugation.
(3) $s_k = \sum_{i=1}^{n} \lambda_i^k \geq 0$.
(4)$s_k^m \leq n^{m-1} s_{km}$ for $k, m = 1, 2, \ldots$ (JLL inequality)[3,8].

A number of necessary conditions for realizability are known, as well as a number of sufficient conditions. In many cases, sufficiency is established by the direct construction of a realizing matrix [1-6].

In terms of n, complete solutions to the NIEP are available only for $n \leq 4$. Nazari and Sherafat in [14] tried to introduce a recursive method for solving (NIEP). They solved different cases for state $n = 5$ and their recursive method can also be used for case $n > 5$. Although they found a nonnegative matrix for many cases of σ, we can say that complete solution for this problem when $n \geq 5$ is an open problem.

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For the case of non-real spectra σ for $n = 4$, complete solutions are available through work of Laffey and Meehan [5](see Meehan s 1998 doctoral thesis (National University of Ireland, Dublin [12])) and, independently, that of Torre-Mayo, Abril-Raymundo, Alarcia-Estevez, Marijuan and Pisanero by analyzing coefficients of the characteristic polynomial. EBL digraphs [11]. Oscar Rojo, Ricardo L. Soto found a necessary and sufficient condition for $\sigma = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ to be the spectrum of some circulant nonnegative matrix [13]. In [10] Helena Šmigoc started with a realizable list of real numbers and obtained a realizable list that contains elements that are not real.

We start by Lemma 6 of paper from Helena Šmigoc in [2] and in section 2 bring Theorem 2.1 from [14] that is similar to Šmigoc's Lemma and find some new condition to solve (NIEP) for a given real list of σ. In section 3 we give some special sets of the spectrum and construct a nonnegative matrix corresponding to them.

Lemma 1.1. *Suppose B is an $m \times m$ matrix with canonical form $J(B)$ that contains at least one 1×1 Jordan block corresponding to the eigenvalue c:*

$$J(B) = \begin{pmatrix} c & 0 \\ 0 & I(B) \end{pmatrix},$$

let t and s, respectively, be the left and right eigenvector of B associated with the 1×1 Jordan block in the above canonical form. Furthermore, we nornmalize vectors t and s so that $t^T s = 1$. Let $J(A)$ be a Jordan canonical form for an $n \times n$ matrix

$$A = \begin{pmatrix} A_1 & a \\ b^T & c \end{pmatrix}$$

where A_1 is an $(n-1) \times (n-1)$ matrix and a and b are vectors in C^{n-1}. Then the matrix

$$C = \begin{pmatrix} A_1 & at^T \\ sb^T & B \end{pmatrix}$$

has Jordan canonical form

$$J(C) = \begin{pmatrix} J(A) & 0 \\ 0 & I(B) \end{pmatrix}.$$

2. Construction of nonnegative matrix with spectrum of two special nonnegative matrices

Theorem 2.1. *Let B be an $m \times m$ nonnegative matrix and $M_1 = \{\mu_1, \mu_2, \ldots, \mu_m\}$ be its eigenvalues and μ_1 be Perron eigenvalue of B. Assume A be an $n \times n$ nonnegative matrix in following form*

$$A = \begin{pmatrix} A_1 & a \\ b^T & \mu_1 \end{pmatrix},$$

where A_1 is an $(n-1) \times (n-1)$ matrix and a and b are arbitrary vectors in C^{n-1} and $M_2 = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ is the set of eigenvalues of A. Then there exist the $(m+n-1) \times (m+n-1)$ nonnegative matrix such that $M = \{\mu_2, \ldots, \mu_m, \lambda_1, \lambda_2, \ldots, \lambda_m\}$ is its eigenvalues.

Proof. Proof in [14]. □

We present a Corollary of the above Theorem, which can be used for problems that do not require the existence of Perrron eigebvalue of the matrix B on the main diagonal of the matrix A.

Corollary 2.2. *Let the conditions of Theorem* 2.1 *satisfy but matrix A is in the following form:*

$$A = \begin{pmatrix} A_1 & a \\ b^T & \alpha\mu_1 \end{pmatrix},$$

then there exists a nonnegative matrix C with the order of $(m+n-1)\times(m+n-1)$ as

$$C = \begin{pmatrix} A_1 & as^* \\ sb^T & \alpha B \end{pmatrix},$$

such that the elements of $M = \{\lambda_1, \dots, \lambda_n, \alpha\mu_2, \dots, \alpha\mu_m\}$ are its eigenvalues.

Proof. Proof is very similar of the Theorem 2.1. □

Remark 2.3. We can use the above Theorem for both symmetric and nonsymmetric matrices. We illustrate this with two examples. It is easy to see that the nonnegative matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 250 & 55 & 2 \end{pmatrix}$ has eigenvalues $\begin{pmatrix} 10 \\ -4-3\,i \\ -4+3\,i \end{pmatrix}$ and the matrix $B = \begin{pmatrix} 0 & 28 \\ 1 & 12 \end{pmatrix}$ has eigenvalues $\begin{pmatrix} 14 \\ -2 \end{pmatrix}$, with normalized Perron eigenvector $s = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$. In order to be able to use the above theorem, we must create a nonnegative matrix whose its Perron eigenvalue is equal 2, because this number is on the main diagonal of matrix A. Now let $a = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $b^T = \begin{pmatrix} 250 & 55 \end{pmatrix}$ then we see that $\frac{1}{7}B$ has eigenvalues $\begin{pmatrix} 2 \\ -2/7 \end{pmatrix}$. Then by above Theorem the nonnegative matrix

$$C = \begin{pmatrix} A_1 & as^* \\ sb^T & \alpha B \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1/4\sqrt{2} & 1/4\sqrt{14} \\ \frac{125\sqrt{2}}{2} & \frac{55\sqrt{2}}{4} & 0 & 2/7\sqrt{7} \\ \frac{125\sqrt{14}}{2} & \frac{55\sqrt{14}}{4} & 2/7\sqrt{7} & \frac{12}{7} \end{pmatrix},$$

is nonnegative matrix with eigenvslues $\begin{pmatrix} -2/7 \\ 10 \\ -4-3\,i \\ -4+3\,i \end{pmatrix}$. It is easy to see that we can use the above Theorem for symmetric matrices. The symmetric matrix $A = \begin{pmatrix} 0 & \sqrt{15} \\ \sqrt{15} & 2 \end{pmatrix}$ has eigenvalues $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$ and we choose the nonnegative 2×2 symmetric matrix as $B = \begin{pmatrix} 0 & \sqrt{14} \\ \sqrt{14} & 12 \end{pmatrix}$ with eigenvalues $\begin{pmatrix} 14 \\ -2 \end{pmatrix}$ and then the matrix $\frac{1}{7}B$ has eigemvalues $\begin{pmatrix} 2 \\ -2/7 \end{pmatrix}$

and the Perron eigenvalue of matrix $\frac{1}{7}B$ lies in main diagonal of matrix A and since the Perron eigenvector of matrix $\frac{1}{7}B$ is $\begin{pmatrix} 1/4\sqrt{2} \\ 1/4\sqrt{14} \end{pmatrix}$, then the following nonnegative symmetric matrix

$$\begin{pmatrix} A_1 & as^T \\ sa^T & \alpha B \end{pmatrix} = \begin{pmatrix} 0 & 1/4\sqrt{15}\sqrt{2} & 1/4\sqrt{15}\sqrt{14} \\ 1/4\sqrt{15}\sqrt{2} & 0 & 2/7\sqrt{7} \\ 1/4\sqrt{15}\sqrt{14} & 2/7\sqrt{7} & \frac{12}{7} \end{pmatrix},$$

has eigenvalues $\begin{pmatrix} 5 \\ -3 \\ -2/7 \end{pmatrix}$.

Corollary 2.4. *If we change the matrices A and C in corollary 2.2, in the following form:*

$$A = \begin{pmatrix} A_1 & a \\ b^T & \alpha + \mu_1 \end{pmatrix}, \; C = \begin{pmatrix} A_1 & as^* \\ sb^T & (\alpha I + B) \end{pmatrix},$$

then the set of $M = \{(\mu_2 + \alpha), \dots, (\mu_m + \alpha), \lambda_1, \dots, \lambda_n\}$ is spectrum of nonnegative matrix C.

Example 2.5. Consider the matrix of Remark (2.3) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 250 & 55 & 1+1 \end{pmatrix}$ with eigenvalues $\begin{pmatrix} 10 \\ -4-3\,i \\ -4+3\,i \end{pmatrix}$, and by above Corolarry $\alpha = \mu_1 = 1$ and $B = \begin{pmatrix} 1/2 & 1/14\sqrt{7} \\ 1/14\sqrt{7} & \frac{13}{14} \end{pmatrix}$ has eigenvalues $\begin{pmatrix} 1 \\ 3/7 \end{pmatrix}$ then $\alpha I + B = \begin{pmatrix} 3/2 & 1/14\sqrt{7} \\ 1/14\sqrt{7} & \frac{27}{14} \end{pmatrix}$ has eigenvalues $\begin{pmatrix} 2 \\ 10/7 \end{pmatrix}$ and the Perron eigenvalue of matrix $\alpha I + B$ lies in main diagonal of matrix A and then

$$C = \begin{pmatrix} A_1 & as^* \\ sb^T & (\alpha I + B) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1/4\sqrt{2} & 1/4\sqrt{14} \\ \frac{125\sqrt{2}}{2} & \frac{55\sqrt{2}}{4} & 3/2 & 1/14\sqrt{7} \\ \frac{125\sqrt{14}}{2} & \frac{55\sqrt{14}}{4} & 1/14\sqrt{7} & \frac{27}{14} \end{pmatrix}$$

has eigenvalues

$$\begin{pmatrix} \frac{10}{7} \\ 10 \\ -4-3\,i \\ -4+3\,i \end{pmatrix}$$

Theorem 2.6. *Assume B is an $m \times m$ nonnegative diagonal matrix and $M_1 = \{\mu_1, \mu_2, \dots, \mu_m\}$ is set of its eigenvalues and μ_i and μ_j are two arbitrary elements of M, without loss of generalization of the problem, let $i = 1$ and $j = 2$. Take A as an $n \times n$ nonnegative matrix as* *follows*

$$A = \begin{pmatrix} A_1 & a_1 & a_2 \\ b_1^T & 0 & 0 \\ b_2^T & 0 & 0 \end{pmatrix},$$

where A_1 is an $(n-2)\times(n-2)$ matrix and a_1, a_2, b_1 and b_2 are arbitrary vectors in C^{n-2}. If $M_2 = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be the set of eigenvalues of A, then there exists a $(3m+n-4)\times(3m+n-4)$ nonnegative matrix, such that $M = \{\mu_3, \dots, \mu_m, \mu_3, \dots, \mu_m, \lambda_1, \dots, \lambda_n, \underbrace{0, \dots, 0}_{m\ times}\}$ is its spectrum.

Proof. Assume vectors s and t are orthonormal eigenvectors associated to eigenvalues μ_1 and μ_2, respectively. By Schur decomposition theorem, there exists the unitary matrix Y such that $Y^*BY = T_B = B$. Now we partition the matrix Y and Y^* in the following form,

$$Y = \begin{pmatrix} s & t & T \end{pmatrix} \qquad and \qquad Y^* = \begin{pmatrix} s^* \\ t^* \\ T^* \end{pmatrix}$$

where T is $m\times(m-2)$ matrix and s and t are $m\times 1$ vectors and it is obvious that,

$$YY^* = ss^* + tt^* + TT^* = I_m,$$

$$Y^*Y = \begin{pmatrix} s^*s & s^*t & s^*T \\ t^*s & t^*t & t^*T \\ T^*s & T^*t & T^*T \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{m-2} \end{pmatrix} \tag{2.5}$$

from the above relation we have,

$$Y^*BY = \begin{pmatrix} \mu_i & 0 & \star \\ 0 & \mu_j & \star \\ 0 & 0 & \hat{T}_B \end{pmatrix} = T_B \tag{2.5}$$

$\hat{T}_B$ is a diagonal matrix with set of $\{\mu_3, \dots, \mu_m\}$ in its main diagonal. By Schur decomposition Theorem, there exist an unitary matrix X, such that $X^*AX = T_A$, is an upper triangular matrix with the elements M_2 in its main diagonal. The matrices X and X^* are partitioned as below,

$$X = \begin{pmatrix} V \\ K \\ L \end{pmatrix} \qquad \text{and} \qquad X^* = \begin{pmatrix} V^* & K^* & L^* \end{pmatrix},$$

where the order of matrix V is $(n-2)\times n$ and the order of K and L are both $1\times n$, since X is a unitary matrix, we have,

$$XX^* = \begin{pmatrix} VV^* & VK^* & VL^* \\ KV^* & KK^* & KL^* \\ LV^* & LK^* & LL^* \end{pmatrix} = \begin{pmatrix} I_{n-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.7}$$
$$X^*X = V^*V + K^*K + LL^* = I_n.$$

By (2.7) and $X^*AX = T_A$, we have,

$$T_A = V^*A_1V + K^*b^TV + L^*b_2^TV + V^*a_1K + V^*a_2L. \tag{2.8}$$

We consider matrices Z and Z^* and nonnegative matrix C with $(3m+n-4)\times(3m+n-4)$ dimension in the following form,

$$Z=\begin{pmatrix} V & 0 & 0 & 0\\ sK & ts^* & 0 & T\\ tL & st^* & T & 0\\ 0 & T^* & 0 & T \end{pmatrix}, Z^*=\begin{pmatrix} V^* & K^*s^* & L^*t^* & 0\\ 0 & st^* & ts^* & T\\ 0 & 0 & T^* & 0\\ 0 & T^* & 0 & 0 \end{pmatrix},$$

$$C=\begin{pmatrix} A_1 & a_1s^* & a_2t^* & 0\\ sb_1^T & TT^*B & 0 & 0\\ tb_2^T & 0 & TT^*B & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using the relations (2.5) and (2.7), it is easy to show that Z is a unitary matrix. Now by the relations (2.5)-(2.8), we can calculate Z^*CZ,

$$Z^*CZ=$$
$$\begin{pmatrix} V^*A_1V+K^*b^TV+L^*b_2^TV+V^*a_1K+V^*a_2L & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & T^*BT & 0\\ 0 & 0 & 0 & T^*BT \end{pmatrix}=$$
$$\begin{pmatrix} T_A & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \hat{T}_B & 0\\ 0 & 0 & 0 & \hat{T}_B \end{pmatrix}=T_C,$$

T_C is an upper triangular matrix and the elements of its main diagonal are the elements of set of M. On the other hand by the relation above C and T_C are similar, then C is the matrix, we were to find, and the proof is completes. □

Corollary 2.7. *Let the conditions of Theorem* 2.5 *satisfy, then there exist* $(3m+n-4)\times(3m+n-4)$ *nonnegative matrix, that*

$$M=\{\alpha\mu_3+\delta,\ldots,\alpha\mu_m+\delta,\beta\mu_3+\delta,\ldots,\beta\mu_m+\delta,\gamma\lambda_1,\ldots,\gamma\lambda_n,\underbrace{0,\ldots,0}_{m\ times}\}$$

is its spectrum, where α, β, δ *and* γ *are arbitrary nonnegative real numbers.*

Proof. The nonnegative matrix C is as the following form:

$$C=\begin{pmatrix} \gamma A_1 & \gamma a_1s^* & \gamma a_2t^* & 0\\ \gamma sb_1^T & \alpha TT^*B+\delta I & 0 & 0\\ \gamma tb_2^T & 0 & \beta T^*TB+\delta I & 0\\ 0 & 0 & 0 & \delta T^*T \end{pmatrix},$$

and it is a solution of the problem. Note that in this Corollary, we construct a unitary matrix for the matrices A, B and C which is the same as Theorem 2.5. □

3. Special cases of NIEP of order n

Theorem 3.1. *Assume* $\sigma=\{\lambda_1,\ldots,\lambda_n\}$*, such that the elements of* σ *are real numbers and* σ *has only one real positive number* λ_1*. Let* σ *satisfies in the following conditions,*

$$\lambda_1+\lambda_2+\ldots+\lambda_n\geq 0. \tag{3.1}$$

Then there exist the nonnegative matrix of order n, such that σ *is its spectrum.*

Proof. Although Suleimanova solved in [7] this problem in 1949 with a companion matrix, we want to find another solution here. We provide proof by induction.. Let $n = 2$, in this case the nonnegative matrix

$$A = \begin{pmatrix} 0 & -\lambda_1\lambda_2 \\ 1 & \lambda_1 + \lambda_2 \end{pmatrix} \tag{3.2}$$

is solution of the problem.

Assume $n = 3$, put $\sigma_1 = \{\lambda_1, \lambda_2\}$, it is clear that σ_1 satisfies in the conditions of theorem, so that σ_1 is spectrum of the matrix (3.2). Let $\sigma_2 = \{\lambda_1 + \lambda_2, \lambda_3\}$, by the relation (3.1), we have,

$$(\lambda_1 + \lambda_2 > 0,\ \lambda_3 \leq 0) \qquad \text{or} \qquad (\lambda_1 + \lambda_2 = 0,\ \lambda_3 = 0), \tag{3.3}$$

then

$$\lambda_1 + \lambda_2 \geq |\lambda_3|. \tag{3.4}$$

The relations above show that σ_2 satisfies in the conditions of theorem and by the case of $n = 2$, σ_2 is the spectrum of 2×2 nonnegative matrix,

$$B = \begin{pmatrix} 0 & -(\lambda_1 + \lambda_2)\lambda_3 \\ 1 & \lambda_1 + \lambda_2 + \lambda_3 \end{pmatrix}.$$

Consequently by (3.4), $\lambda_1 + \lambda_2$ is Perron eigenvalue of nonnegative matrix B. It is easy to show that the right orthonormal eigenvector associated with the Perron eigenvalue of B is

$$s = \begin{pmatrix} \frac{-\lambda_3}{\sqrt{1+\lambda_3^2}} \\ \frac{1}{\sqrt{1+\lambda_3^2}} \end{pmatrix},$$

because of the Perron eigenvalue of B is placed on the main diagonal of nonnegative matrix A, then matrices A and B satisfy in theorem 2.1, therefore the nonnegative matrix

$$C = \begin{pmatrix} 0 & \frac{\lambda_1\lambda_2\lambda_3}{\sqrt{1+\lambda_3^2}} & \frac{-\lambda_1\lambda_2}{\sqrt{1+\lambda_3^2}} \\ \frac{-\lambda_3}{\sqrt{1+\lambda_3^2}} & 0 & -(\lambda_1 + \lambda_2)\lambda_3 \\ \frac{1}{\sqrt{1+\lambda_3^2}} & 1 & \lambda_1 + \lambda_2 + \lambda_3 \end{pmatrix} \tag{3.5}$$

has spectrum of $\sigma = \{\lambda_1, \lambda_2, \lambda_3\}$.

Now assume problem holds for $n - 1$, in order to construct a $n \times n$ nonnegative matrix with the set of eigenvalues $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, we have to do the following process.

Let $\sigma_1 = \{\lambda_1, \lambda_2, \ldots, \lambda_{n-1}\}$ and $\lambda = \lambda_1 + \lambda_2 + \ldots, \lambda_{n-2}$ then by relation (3.1) we have

$$(\lambda > 0\ , \lambda_{n-1} \leq 0) \qquad \text{or} \qquad (\lambda = 0,\ \lambda_{n-1} = 0), \text{and then we have}$$
$$\lambda \geq |\lambda_{n-1}|.$$

So that, σ_1 satisfies in the conditions of our theorem, by the hypothesis of induction we can construct the $(n-1) \times (n-1)$ nonnegative matrix A with spectrum of σ_1. By (3.2) and (3.5), the nonnegative matrix A is as the following form

$$A = \begin{pmatrix} A_1 & a \\ b^T & \lambda + \lambda_{n-1} \end{pmatrix},$$

where A_1 is $(n-2) \times (n-2)$ matrix and a and b are the vectors with dimension of $(n-1) \times 1$. Let $\lambda' = \lambda + \lambda_{n-1}$, by (3.1) we have $\lambda' \geq |\lambda_n|$, then by the case of $n = 2$ there exist the 2×2 nonnegative matrix B, with spectrum $\sigma_2 = \{\lambda', \lambda_n\}$ in the following form

$$B = \begin{pmatrix} 0 & -\lambda'\lambda_n \\ 1 & \lambda' + \lambda_n \end{pmatrix}.$$

It is clear that λ' is Perron eigenvalue of nonnegative matrix B and orthonormal eigenvector corresponding to λ' is

$$s = \begin{pmatrix} \frac{-\lambda_n}{\sqrt{1+\lambda_n^2}} \\ \frac{1}{\sqrt{1+\lambda_n^2}} \end{pmatrix}.$$

The nonnegative matrices A and B are satisfied on theorem 2.1. then the nonnegative matrix,

$$C = \begin{pmatrix} A_1 & as^* \\ sb^T & B \end{pmatrix},$$

with spectrum $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is a solution of our problem. □

Theorem 3.2. *Assume $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and $n \geq 3$. If λ_1 be positive real number and λ_2 and λ_3 be pair complex numbers and the other elements of σ be negative or zero real numbers and the conditions* (3.1) *and* (3.2) *satisfy and furthermore elements of σ satisfy in the following condition,*

$$\alpha_1 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + |\lambda_2|^2 \leq 0.$$

Then there exists an $n \times n$ nonnegative matrix such that σ is its spectrum.

Proof. We prove by induction on n. For $n = 3$, the following nonnegative matrix,

$$A = \begin{pmatrix} 0 & \lambda_1\lambda_2\lambda_3 & 0 \\ 0 & 0 & 1 \\ 0 & -\alpha_1 & \lambda_1 + \lambda_2 + \lambda_3 \end{pmatrix},$$

is a solution of the problem.
Assume the proposition for $n-1$ satisfies, then so as to construct on $n \times n$ nonnegative matrix with spectrum $\sigma = \{\lambda_1, \ldots, \lambda_n\}$, we can use the process of Theorem 3.1. □

Theorem 3.3. *Assume $\sigma = \{\lambda_1, \ldots, \lambda_n\}$, such that σ has only one negative number λ_2 and other elements of it, are nonnegative real numbers, furthermore assume the conditions* (3.1) *and* (3.2) *satisfy, then there exists an $n \times n$ nonnegative matrix, such that σ is its spectrum.*

Proof. If $n = 2$, the 2×2 nonnegative matrix (3.3) is a solution of this problem. If $n > 2$, we consider the following matrix,

$$C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where A is the matrix of (3.3) and B is nonnegative diagonal matrix of $n - 2$ order in the following form,

$$B = \operatorname{diag}(\lambda_3, \ldots, \lambda_n)$$

consequently C is solution of the problem. □

Example 3.4. $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, $\mu = \sqrt{1 + \lambda_3^2}$, $\beta = \sqrt{1 + \lambda_4^2}$

$$A_1 = \begin{pmatrix} 0 & \frac{\lambda_1\lambda_2\lambda_3}{\mu} & \frac{\lambda_1\lambda_2\lambda_4}{\mu\beta} & \frac{-\lambda_1\lambda_2}{\mu\beta} \\ \frac{-\lambda_3}{\mu} & 0 & \frac{(\lambda_1+\lambda_2)\lambda_3\lambda_4}{\beta} & \frac{-(\lambda_1+\lambda_2)\lambda_3}{\beta} \\ \frac{-\lambda_4}{\mu\beta} & \frac{-\lambda_4}{\beta} & 0 & -(\lambda_1 + \lambda_2 + \lambda_3)\lambda_4 \\ \frac{1}{\mu\beta} & \frac{1}{\beta} & 1 & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \end{pmatrix}$$

$\sigma_2 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}, \gamma = \sqrt{1+\lambda_5^2}$

$$A_2 = \begin{pmatrix} 0 & \frac{\lambda_1\lambda_2\lambda_3}{\mu} & \frac{\lambda_1\lambda_2\lambda_4}{\mu\beta} & \frac{\lambda_1\lambda_2\lambda_5}{\mu\beta\gamma} & \frac{-\lambda_1\lambda_2}{\mu\beta\gamma} \\ \frac{-\lambda_3}{\mu} & 0 & \frac{(\lambda_1+\lambda_2)\lambda_3\lambda_4}{\beta} & \frac{(\lambda_1+\lambda_2)\lambda_3\lambda_5}{\beta\gamma} & \frac{-(\lambda_1+\lambda_2)\lambda_3}{\beta\gamma} \\ \frac{-\lambda_4}{\mu\beta} & \frac{-\lambda_4}{\beta} & 0 & \frac{(\lambda_1+\lambda_2+\lambda_3)\lambda_4\lambda_5}{\gamma} & \frac{-(\lambda_1+\lambda_2+\lambda_3)\lambda_4}{\gamma} \\ \frac{-\lambda_5}{\mu\beta\gamma} & \frac{-\lambda_5}{\beta\gamma} & \frac{-\lambda_5}{\gamma} & 0 & -(\lambda_1+\lambda_2+\lambda_3+\lambda_4)\lambda_5 \\ \frac{1}{\mu\beta\gamma} & \frac{1}{\beta\gamma} & \frac{1}{\gamma} & 1 & \lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5 \end{pmatrix},$$

$\sigma_3 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}, \eta = \sqrt{1+\lambda_5^2},\ \lambda = \lambda_1+\lambda_2+\lambda_3,\ \lambda' = \lambda+\lambda_4+\lambda_5$

$$A_3 = \begin{pmatrix} 0 & \frac{\lambda_1\lambda_2\lambda_3}{\mu} & \frac{\lambda_1\lambda_2\lambda_4}{\mu\beta} & \frac{\lambda_1\lambda_2\lambda_5}{\mu\beta\gamma} & \frac{\lambda_1\lambda_2\lambda_6}{\mu\beta\gamma\eta} & \frac{-\lambda_1\lambda_2}{\mu\beta\gamma\eta} \\ \frac{-\lambda_3}{\mu} & 0 & \frac{(\lambda_1+\lambda_2)\lambda_3\lambda_4}{\beta} & \frac{(\lambda_1+\lambda_2)\lambda_3\lambda_5}{\beta\gamma} & \frac{(\lambda_1+\lambda_2)\lambda_3\lambda_6}{\beta\gamma\eta} & \frac{-(\lambda_1+\lambda_2)\lambda_3}{\beta\gamma\eta} \\ \frac{-\lambda_4}{\mu\beta} & \frac{-\lambda_4}{\beta} & 0 & \frac{\lambda\lambda_4\lambda_5}{\gamma} & \frac{\lambda\lambda_4\lambda_6}{\gamma\eta} & \frac{-\lambda\lambda_4}{\gamma\eta} \\ \frac{-\lambda_5}{\mu\beta\gamma} & \frac{-\lambda_5}{\beta\gamma} & \frac{-\lambda_5}{\gamma} & 0 & \frac{(\lambda+\lambda_4)\lambda_5\lambda_6}{\eta} & \frac{-(\lambda+\lambda_4)\lambda_5}{\eta} \\ \frac{-\lambda_6}{\mu\beta\gamma\eta} & \frac{-\lambda_6}{\beta\gamma\eta} & \frac{-\lambda_6}{\gamma\eta} & \frac{-\lambda_6}{\eta} & 0 & -\lambda'\lambda_6 \\ \frac{1}{\mu\beta\gamma\eta} & \frac{1}{\beta\gamma\eta} & \frac{1}{\gamma\eta} & \frac{1}{\eta} & 1 & \lambda'+\lambda_6 \end{pmatrix}.$$

We select the next example from [15] and try to find a nonsymmetric nonnegative matrix for the given σ.

Example 3.5. Assume given

$$\sigma = \{\lambda_1 = 15, \lambda_2 = -1, \lambda_3 = -2, \lambda_4 = -3, \lambda_5 = -4, \lambda_6 = -5\},$$

since $\lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \leq 0$ and $\Sigma_{i=1}^6 \lambda_i \geq 0$ then by Theorem (3.1) we construct a solution for σ. At first it is easy to see that the symmetric matrix $C_1 = \begin{pmatrix} 0 & 15 \\ 1 & 14 \end{pmatrix}$ has eigenvalues $\begin{pmatrix} 15 \\ -1 \end{pmatrix}$ and the matrix $B = \begin{pmatrix} 0 & 28 \\ 1 & 12 \end{pmatrix}$ has eigenvalues $\begin{pmatrix} 14 \\ -2 \end{pmatrix}$, with normalized Perron eigenvector $s = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$, then let $a = \begin{pmatrix} 15 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \end{pmatrix}$ then the 3×3 following matrix

$$C_2 = \begin{pmatrix} A_2 & as^* \\ sb^T & B \end{pmatrix} = \begin{pmatrix} 0 & 6\sqrt{5} & 3\sqrt{5} \\ 2/5\sqrt{5} & 0 & 28 \\ 1/5\sqrt{5} & 1 & 12 \end{pmatrix},$$

has eigenvalues

$$\begin{pmatrix} 15 \\ -2 \\ -1 \end{pmatrix},$$

and it is necessary to mention that the member $(3,3)$ of the matrix C_1 is equal to 12, and this makes it possible to continue the algorithm. Again the matrix $B = \begin{pmatrix} 0 & 36 \\ 1 & 9 \end{pmatrix}$ has eigenvalues $\begin{pmatrix} 12 \\ -3 \end{pmatrix}$ with normalazed Perron eigenvaector $s = \begin{pmatrix} 3/10\sqrt{10} \\ 1/10\sqrt{10} \end{pmatrix}$, in this case

$a = \begin{pmatrix} 3\sqrt{5} \\ 28 \end{pmatrix}$ and $b = \begin{pmatrix} 1/5\sqrt{5} \\ 1 \end{pmatrix}$ so that

$$C_3 = \begin{pmatrix} A_2 & as^* \\ sb^T & B \end{pmatrix} = \begin{pmatrix} 0 & 6\sqrt{5} & 9/2\sqrt{2} & 3/2\sqrt{2} \\ 2/5\sqrt{5} & 0 & \frac{42}{5}\sqrt{10} & \frac{14}{5}\sqrt{10} \\ 3/10\sqrt{2} & 3/10\sqrt{10} & 0 & 36 \\ 1/10\sqrt{2} & 1/10\sqrt{10} & 1 & 9 \end{pmatrix},$$

has eigenvalues

$$\begin{pmatrix} 15 \\ -3 \\ -2 \\ -1 \end{pmatrix}.$$

With continue this method we have

$$C_4 = \begin{pmatrix} 0 & 6\sqrt{5} & 9/2\sqrt{2} & \frac{6}{17}\sqrt{2}\sqrt{17} & \frac{3}{34}\sqrt{2}\sqrt{17} \\ 2/5\sqrt{5} & 0 & \frac{42}{5}\sqrt{10} & \frac{56}{85}\sqrt{10}\sqrt{17} & \frac{14}{85}\sqrt{10}\sqrt{17} \\ 3/10\sqrt{2} & 3/10\sqrt{10} & 0 & \frac{144}{17}\sqrt{17} & \frac{36}{17}\sqrt{17} \\ \frac{2}{85}\sqrt{2}\sqrt{17} & \frac{2}{85}\sqrt{10}\sqrt{17} & \frac{4}{17}\sqrt{17} & 0 & 36 \\ \frac{1}{170}\sqrt{2}\sqrt{17} & \frac{1}{170}\sqrt{10}\sqrt{17} & 1/17\sqrt{17} & 1 & 5 \end{pmatrix}$$

with eigenvalues

$$\begin{pmatrix} 15 \\ -4 \\ -3 \\ -2 \\ -1 \end{pmatrix}$$

and finally with round the solution with 4 floating point, we have

$$C_5 = \begin{pmatrix} 0.0 & 13.42 & 6.363 & 2.057 & 0.5047 & 0.1009 \\ 0.8944 & 0.0 & 26.56 & 8.591 & 2.106 & 0.4212 \\ 0.4242 & 0.9486 & 0.0 & 34.93 & 8.559 & 1.712 \\ 0.1372 & 0.3068 & 0.9701 & 0.0 & 35.30 & 7.062 \\ 0.03364 & 0.07519 & 0.2377 & 0.9805 & 0.0 & 25.0 \\ 0.006729 & 0.01504 & 0.04755 & 0.1961 & 1.0 & 0.0 \end{pmatrix}$$

References

[1] T.J. Laffey, Helena Šmigoc, On a classic example in the nonnegative inverse eigenvalue problem, ELA, Volume 17, pp. 333–342, July 2008.

[2] Helena Šmigoc, The inverse eigenvalue problem for nonnegative matrices, Linear Algebra and its Applications 393(2004)365–374.

[3] R. Lowey, D. London, A note on an inverse problem for nonnegative matrices, Linear and Multilinear Algebra 6(1978)83–90.

[4] R. Reams, An inequlity for nonnegative matrices and the inverse eigenvalue problem, Linear and Multilinear Algebra 41(1996)367-375.
[5] T.J. Laffey, E. Meehan, A characterization of trace zero nonnegative 5×5 matrices, Linear Algebra Appl. 302–303(1999)295–302.
[6] Kh. D. Ikramov, V. N. Chugunov, Inverse matrix eigenvalue problems, Journal of Mathematical Sciences, Vol. 98, No. 1, 2000.
[7] K. R. Suleimanova, Stochastic matrices with real characteristic values, Dokl. Akad. Nauk. SSSR, 66(1949), 343–345.
[8] C.R. Johnson, Row stochastic matrices similar to doubly stochastic matrices, Linear and Multilinear Algebra 10 (2) (1981) 113-130.
[9] S. Friedland, On an inverse problem for nonnegative and eventually nonnegative matrices, Israel J. Math. 29 (1) (1978) 43-60.
[10] Helena Šmigoc, Construction of nonnegative matrices and the inverse eigenvalue problem, Linear and Multilinear Algebra, Vol. 53, No. 2, March 2005, 85-96.
[11] J. Torre-Mayo, M.R. Abril-Raymundo, E. Alarcia-Estevez, C. Marijuan, M. Pisonero, The nonnegative inverse eigenvalue problem from the coefficients of the characteristic polynomial. EBL digraphs, Linear Algebra and its Applications 426 (2007) 729-773.
[12] Eleanor Meehan, Some Results On Matrix Spectra. PhD thesis, University College Dublin, 1998.
[13] Oscar Rojo, Ricardo L. Soto, Existence and construction of nonnegative matrices with complex spectrum, Linear Algebra and its Applications 368 (2003) 53-69.
[14] A. M. Nazari, F. Sherafat, On the inverse eigenvalue problem for nonnegative matrices of order two to five. Linear Algebra and its Applications 436 (2012) 1771-1790.
[15] A.M. Nazari, A. Mashayekhi, A. Nezami, On the inverse eigenvalue problem of symmetric nonnegative matrices, *Mathematical Sciences*, volume 14, (2020) 11-19.

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