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Research Paper

NUMERICAL INVESTIGATION OF WAVE EQUATIONS IN LARGE DOMAINS VIA A NOVEL VARIATIONAL ITERATION METHOD

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Abstract. The value of an auxiliary parameter incorporated into the well-known variational iteration method (VIM) to obtain solutions of wave equations in unbounded domains is discussed in this article. The suggested method, namely the optimal variational iteration method, is investigated for convergence. Furthermore, the proposed method is tested on one-dimensional and two-dimensional wave equations in unbounded domains in order to better understand the solution mechanism and choose the best auxiliary parameter.Comparisons with results from the standard variational iteration procedure demonstrate that the auxiliary parameter is very useful in tracking the convergence field of the approximate solution.

Keywords: Wave equations, Unbounded domaines, Variational iteration method, Optimal variational iteration method, Auxiliary parameter, Hermite-Gauss quadrature.

1. Introduction

Nonlinear partial differential equations can be used to describe a wide range of phenomena in a variety of fields. Most of these equations do not have a precise analytical solution, so these nonlinear equations should be solved by approximate methods. Since most of these equations do not have a direct empirical solution, they must be solved using approximate methods. Wave equations are the most important of these equations, so we concentrated our research on them. Waves appear in unbounded media in a variety of areas, including vibrations, aerodynamics, solid geophysics, oceanography, meteorology, and electromagnetics.
In unbounded domains, the wave equation is as follows:

$$\frac{\partial^2 \mathbf{u}(\mathbf{x},t)}{\partial t^2} - c^2 \triangle \mathbf{u}(\mathbf{x},t) = 0, \tag{1.1}$$

subject to the initial conditions

$$\begin{cases} \mathbf{u}(\mathbf{x},0) = f(\mathbf{x}) \\ \mathbf{u}_t(\mathbf{x},0) = g(\mathbf{x}), \end{cases} \qquad \mathbf{x} \in \mathbb{R}^n. \tag{1.2}$$

Since the 1970s, numerical approaches for such problems have been developed[15]. The key categories of methods that have appeared are boundary integral methods, infinite element methods, absorbing layer methods, and non-reflecting boundary state (NRBC) methods [13, 23, 24, 25, 26, 27, 28, 29]. The standard method for solving such problems numerically

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is to create an imaginary boundary and add acceptable boundary conditions [2, 14, 20, 19]. The process of integral equations is also a useful technique for converting this partial differential equation to an integral equation on the scatterer's bounded surface [9, 34].Laplace-Fourier methods with Galerkin boundary elements in space and collocation methods with some stabilization techniques are some of the latest numerical methods [3, 5, 6, 7, 17, 18]. In Ref.[8] a quick version of the marching-on-in-time (MOT) approach is proposed, which is based on a suitable plane wave expansion of the arising potential, resulting in lower computational and storage costs [1]. Wazwaz used the Adomian decomposition approach to present approximate analytical solutions to wave equations in the unbounded domain. Furthermore, the Homotopy perturbation technique is used to approximate wave equation solutions in an unbounded domain [40, 4].

The variational iteration method is a more efficient and convenient analytical procedure. In Ref. [21], Ji-Huan He is a Chinese mathematician. He presented a very clear and basic explanation of the variational iteration process, which was further developed by the inventor himself. [39, 22, 37]. The method's key feature is its adaptability and ability to obtain reliable and convenient solutions to nonlinear equations [38, 16, 33, 10]. Furthermore, there are many variations of the variational iteration process, the most appealing of which is Herisanu and Marinca's, in which the variational iteration method is combined with least squares technology, and one iteration leads to perfect outcomes [22]. Yilmaz and Inc developed a variational iteration algorithm with an auxiliary parameter to change the convergence rate, but they did not have a general rule for the right auxiliary parameter selection[10]. Hosseini et al. improved this updated approach by adding several profitable rules for determining the auxiliary parameter optimally [11, 35, 32, 12].

The variational iteration approach with an auxiliary parameter, optimum variational, is successfully used in the present paper to achieve approximate solutions of wave equations in unbounded domains. The residual function and the residual function error are described in the proposed method to select the auxiliary parameter optimally.

2. **The variational iteration method**

The basic solution protocol of the variational iteration process is briefly recapitulated here. Consider the following equation in terms of function:

$$Hu = Lu + Nu + Ru + g(x), \tag{2.1}$$

where L represents the higher order derivative, which is thought to be conveniently invertible. R stands for a linear differential operator with a lower order than L, Nu for nonlinear terms, and g for the root name. He′s approach is characterized by the construction of a correction functional for the equation **(2.1)**, which reads as follows:

$$u_{n+1}\left(x\right) = u_n\left(x\right) + \int_0^x \lambda\left(s\right) Hu_n\left(s\right) ds, \tag{2.2}$$

where λ is a Lagrange multiplier which can be identified optimally via variational theory [39], u_n is the nth approximate solution, and $\tilde{u}_n$ denotes a restricted variation, i.e., $\partial \tilde{u}_n = 0$. After identification of the multiplier, a variational iteration algorithm is constructed, an exact solution can be achieved when n tends to infinite:

$$u(x) = \lim_{n\to\infty} u_n(x). \tag{2.3}$$

In summary, we have the following variational iteration formula for **(2.1)**:

$$\begin{cases} u_0(x) \; is \; an \; arbitrary \; function, \\ u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s) \, H u_n(s) \, ds, \qquad n \geqslant 0. \end{cases} \tag{2.4}$$

3. Optimal Variational Iteration Method

In the variational iteration procedure, equation **(2.4)**, an unknown auxiliary parameter can be included:

$$\begin{cases} u_0(x) \; is \; an \; arbitrary \; function, \\ u_1(x,h) = u_0(x) + h \int_0^x \lambda(s) \, H u_n(s) \, ds, \\ u_{n+1}(x,h) = u_n(x,h) + h \int_0^x \lambda(s) \, H u_n(s,h) \, ds, \qquad n \geqslant 1. \end{cases} \tag{3.1}$$

It should be noted that symbolic computation tools like Maple or Mathematica can compute $u_{n+1}(x,h)$, $n \geqslant 1$. The auxiliary parameter h is included in the approximate solutions $u_{n+1}(x,h)$, $n \geqslant 1$. The method's validity is based on the premise that the approximation $u_{n+1}(x,h)$, $n \geqslant 1$, converges to the exact solution. It is the auxiliary parameter that guarantees the assumption is met. In general, using the residual function's error of norm two, it is simple to select an appropriate value of h that assures the approximation solutions are convergent [10, 11, 35, 32]. In reality, the suggested technique is relatively basic, easier to implement, and capable of more correctly approximating the solution in a broad solution domain.

4. Convergence analysis

In this section, the convergence of the optimal variational iteration method is studied according to the alternative approach of this method which present in the following.
This approach can be implemented, in a reliable and efficient way, to handle the nonlinear differential equation **(2.1)**. The linear operator L is defined as $L = \frac{\partial^2}{\partial t^2}$, when the optimal variational iteration method is applied to solve the wave equation **(1.1)**.
Now, define the operator A as,

$$Au(\mathbf{x},t,h) = h \int_0^t \lambda(\tau) \, H u(\mathbf{x},\tau,h) d\tau, \tag{4.1}$$

and define the components v_n, s_n, $n \geq 0$, as,

$$\begin{cases} v_0(\mathbf{x},t) = u_0(\mathbf{x},t), \\ s_0(\mathbf{x},t) = v_0(\mathbf{x},t), \end{cases}$$

$$\begin{cases} v_1(\mathbf{x},t,h) = As_0(\mathbf{x},t), \\ s_1(\mathbf{x},t,h) = s_0(\mathbf{x},t) + v_1(\mathbf{x},t,h), \end{cases}$$

and in general for $n \geq 1$,

$$\begin{cases} v_{n+1}(\mathbf{x},t,h) = As_n(\mathbf{x},t,h), \\ s_{n+1}(\mathbf{x},t,h) = s_n(\mathbf{x},t,h) + v_{n+1}(\mathbf{x},t,h), \end{cases} \tag{4.2}$$

then, consequently, we have,

$$u(\mathbf{x},t,h) = \lim_{n\to\infty} s_n(\mathbf{x},t,h) = v_0(\mathbf{x},t) + \sum_{n=1}^{\infty} v_n(\mathbf{x},t,h). \tag{4.3}$$

The zeroth approximation $u_0(\mathbf{x},t)$ can be freely chosen if it satisfies the initial conditions of the problem and $Lu_0(\mathbf{x},t)=0$. For the approximation purpose, we approximate the solution $u(\mathbf{x},t,h)=v_0(\mathbf{x},t)+\sum_{n=1}^{\infty}v_n(\mathbf{x},t,h)$, by the Nth-order truncated series $u_N(\mathbf{x},t,h)=v_0(\mathbf{x},t)+\sum_{n=1}^{N}v_n(\mathbf{x},t,h)$.
The approximate solutions $u_N(t,h)$, contains the auxiliary parameter h. It is the auxiliary parameter that ensures that the assumption can be satisfied, in general, by means of the error of norm two of the residual function.
The sufficient conditions for convergence of the method and the error estimate will be introduced. The main results are proposed in the following theorems [36].

Theorem 4.1 Let A, defined in **(4.1)**, be an operator from a Hilbert space H to H. If $\exists\tilde{h}\neq 0, 0<\gamma<1$, such that,

$$\begin{cases} \parallel As_0(\mathbf{x},t)\parallel\ \leq\gamma\parallel s_0(\mathbf{x},t)\parallel, & \\ \parallel As_1(\mathbf{x},t,\tilde{h})\parallel\leq\gamma\parallel As_0(\mathbf{x},t)\parallel, & \\ \parallel As_n(\mathbf{x},t,\tilde{h})\parallel\leq\gamma\parallel As_{n-1}(\mathbf{x},t,\tilde{h})\parallel, & n=2,3,4,\cdots. \end{cases}$$

Then the series solution defined in **(4.3)**,

$$u(\mathbf{x},t)=\lim_{n\to\infty}s_n(\mathbf{x},t,\tilde{h})=v_0(\mathbf{x},t)+\sum_{n=1}^{\infty}v_n(\mathbf{x},t,\tilde{h}),$$

converges [11].

Lemma 4.1 Let L, defined in **(2.1)**, be as follow as, $L=\dfrac{\partial^2}{\partial t^2}$, and λ identified optimally via variational theory [11]. If k, be a function from a Hilbert space H to H, then we have,

$$L\left\{\int_0^t\lambda(\tau)\,k(\mathbf{x},\tau)d\tau\right\}=-k(\mathbf{x},t).$$

Theorem 4.2 Let L, defined in **(2.1)**, be as follow as, $L=\dfrac{\partial^2}{\partial t^2}$, if we have $u(\mathbf{x},t)=v_0(\mathbf{x},t)+\sum_{n=1}^{\infty}v_n(\mathbf{x},t,\tilde{h})$, then $u(\mathbf{x},t)$, is an exact solution of the nonlinear problem **(2.1)**.

Theorem 4.3 Suppose that the series solution $u(\mathbf{x},t)=v_0(\mathbf{x},t)+\sum_{n=1}^{\infty}v_n(\mathbf{x},t,\tilde{h})$, defined in **(4.3)**, is convergent to exact solution of the nonlinear problem **(2.1)**. If the truncated series $u_N(\mathbf{x},t)=v_0(\mathbf{x},t)+\sum_{n=1}^{N}v_n(\mathbf{x},t,\tilde{h})$, is used as an approximate solution, then the maximum error is estimated as,

$$\parallel u(\mathbf{x},t)-u_N(\mathbf{x},t)\parallel\ \leq\ \frac{1}{1-\gamma}\gamma^{N+1}\parallel v_0\parallel.$$

In summary, we can define,

$$\beta_i=\begin{cases}\dfrac{\parallel v_{i+1}\parallel}{\parallel v_i\parallel}, & \parallel v_i\parallel\neq 0,\\ 0, & \parallel v_i\parallel=0,\end{cases}\qquad i=0,1,2,\cdots.$$

Now, if $0<\beta_i<1$ for $i=0,1,2,\cdots$, then the series solution $v_0(\mathbf{x},t)+\sum_{n=1}^{\infty}v_n(\mathbf{x},t,\tilde{h})$, of problem **(2.1)** converges to an exact solution, $u(\mathbf{x},t)$. Moreover, as stated in Theorem 4.3, the maximum absolute truncation error is estimated to be,

$$\| u(\mathbf{x},t) - u_N(\mathbf{x},t) \| \le \frac{1}{1-\beta}\beta^{N+1} \| v_0 \|,$$

where $\beta = max\{\beta_i, i = 0,1,2,\cdots\}$.

Notice that, the first finite terms do not affect the convergence of series solution. In other words, if the first finite β_i's, $i = 0,1,2,\cdots,l$, are not less than one and $\beta_i < 1$, for $i > l$, then, of course the series solution $v_0(\mathbf{x},t) + \sum_{n=1}^{\infty} v_n(\mathbf{x},t,\tilde{h})$, of problem **(2.1)**, converges to an exact solution [36].

5. Numerical Examples

To elucidate the solution procedure, three examples are given. In each case, the standard variational iteration method is applied on the wave equations in unbounded domains. The obtained results show that where the solutions of wave equations in unbounded domains are under investigation, the standard variational iteration method is not applicable. Therefore, the proposed method is tested on the aforementioned wave equation in unbounded domains. Comparison with results by exact solutions indicates that the large domains will not decrease the effectiveness of the proposed method.

Example 5.1 Consider the following wave equation [38]:

$$\begin{cases} u_{tt} = u_{xx}, & & -\infty < x < \infty, & t > 0, \\ u(x,0) = \sin(x), & u_t(x,0) = 0, & -\infty < x < \infty, & \end{cases} \tag{5.1}$$

which admits the solution $u(x,t) = \sin(x)\cos(t)$. Take $(x,t) \in (-\infty,\infty) \times [0,50]$. According to the standard VIM we have the following variational iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t)\left\{\frac{\partial^2 u_n(x,s)}{\partial s^2} - \frac{\partial^2 u_n(x,s)}{\partial x^2}\right\} ds. \tag{5.2}$$

Beginning with $u_0(x,t) = u(x,0) + tu_t(x,0) = \sin(x)$, we stop the solution procedure at $u_{60}(x,t)$. Figure **1**, is the absolute error of $u_{60}(x,t)$, for $(x,t) \in [-10^8, 10^8] \times [0,50]$, showing that the solution $u_{60}(x,t)$ is not valid for large values of x and t, of course, the accuracy can be improved if the iteration procedure continues and the exact solution can be obtained when n tends to infinite. Now, using the recursive scheme (**3.1**), we successively have:

$$\begin{aligned} u_0(x,t) &= u(x,0) + tu_t(x,0) = \sin(x), \\ u_1(x,t) &= \sin(x) - \tfrac{1}{2}ht^2\sin(x), \end{aligned}$$

and in general,

$$\begin{aligned} &u_{n+1}(x,t,h) = u_n(x,t,h) \\ &+h\int_0^t (s-t)\left\{\frac{\partial^2 u_n(x,s,h)}{\partial s^2} - \frac{\partial^2 u_n(x,s,h)}{\partial x^2}\right\} ds, \qquad n \geqslant 1. \end{aligned} \tag{5.3}$$

In order to find a proper value of h for the approximate solutions (**5.3**), we define the following residual function,

$$r_{60}(x,t,h) = \frac{\partial^2 u_{60}(x,t,h)}{\partial t^2} - \frac{\partial^2 u_{60}(x,t,h)}{\partial x^2}, \tag{5.4}$$

and the following error of residual function,

$$e_{60}(h) = \int_{-\infty}^{\infty}\int_{0}^{50} |r_{60}(x,t,h)|^2 \, dt \, dx. \tag{5.5}$$

We apply Hermite-Gauss quadrature as a numerical integration to calculate the approximate $e_{60}(h)$ [31]. For obtaining an optimal value of h, we choose the minimum point of the error residual function (**5.4**). The minimum point of $e_{60}(h)$, as $h = 0.95077$, is obtained by using optimization package of Maple software in 2 seconds. By substituting $h = 0.95077$, in $u_{60}(x,t,h)$, the absolute error of the 60th-order approximation of the proposed method reduces remarkably, as shown in Figure **2**.

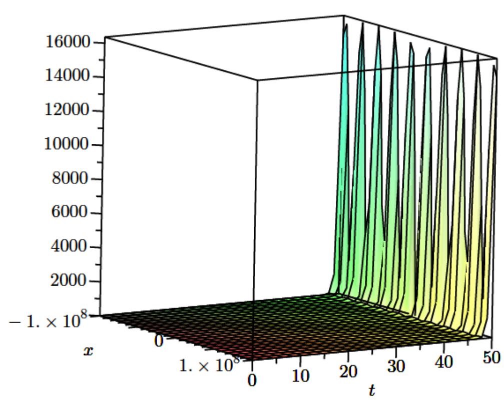

FIGURE 1. Absolute error for the 60th-order approximation by standard VIM for $u(x,t)$ in example 5.1.

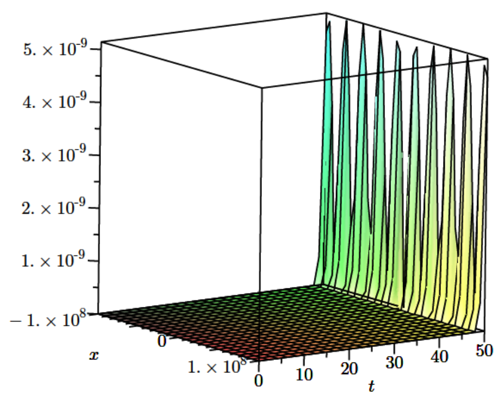

FIGURE 2. Absolute error for the 60th-order approximation by optimal VIM when $h = 0.95077$, in example 5.1.

Example 5.2 Consider the following wave equation [38]:

$$\begin{cases} u_{tt} = u_{xx}, & -\infty < x < \infty, \quad t > 0. \\ u(x,0) = \sin(x), \quad u_t(x,0) = \cos(x), & -\infty < x < \infty, \end{cases} \tag{5.6}$$

It is easy to verify that $u(x,t) = \sin(x+t)$. We take the solution domain as $(x,t) \in (-\infty,\infty) \times [0,100]$. Similarly the absolute error of $u_{120}(x,t)$ for $(x,t) \in [-10^8, 10^8] \times [0,100]$, tends to very large values, when time tends to 100, as illustrated in Figure **3**. Using the iteration formulation (**3.1**), we successively have

$$u_0(x,t) = u(x,0) + t u_t(x,0) = \sin(x) + t\cos(x),$$
$$u_1(x,t) = \sin(x) + t\cos(x) + \tfrac{1}{6}h\left(t^3\cos(x) + 3t^2\sin(x)\right),$$

and in general,

$$\begin{aligned} &u_{n+1}(x,t,h) = u_n(x,t,h) \\ &+h\int_0^t (s-t)\left\{\frac{\partial^2 u_n(x,s,h)}{\partial s^2} - \frac{\partial^2 u_n(x,s,h)}{\partial x^2}\right\} ds, \quad n \geqslant 1. \end{aligned} \tag{5.7}$$

We define the residual function of $u_{120}(x,t)$ as,

$$r_{120}(x,t,h) = \frac{\partial^2 u_{120}(x,t,h)}{\partial t^2} - \frac{\partial^2 u_{120}(x,t,h)}{\partial x^2}. \tag{5.8}$$

For obtaining an optimal value of h, we choose the global minimum point of the error of residual function (**5.8**):

$$e_{120}(h) = \int_{-\infty}^{\infty}\int_{0}^{100} |r_{120}(x,t,h)|^2 \, dt \, dx. \tag{5.9}$$

Thus, we select $h = 0.93258$, and it's absolute error is reduced greatly for $u_{120}(x,t), (x,t) \in [-10^8, 10^8] \times [0,100]$, as shown in Figure **4**.

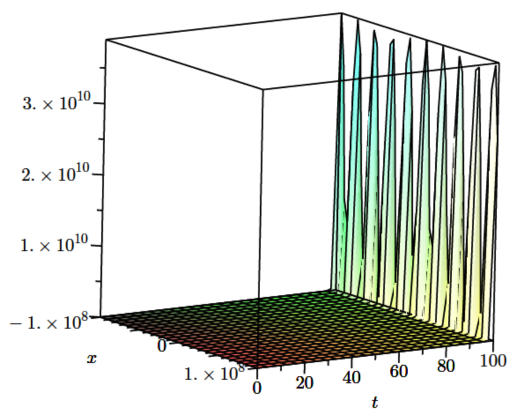

FIGURE 3. Absolute error for the 120th-order approximation by standard VIM for $u(x,t)$ in example 5.2.

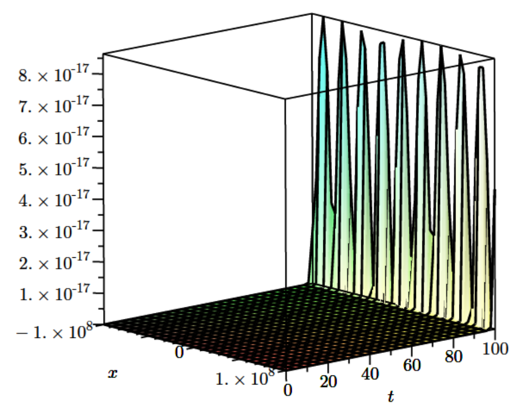

FIGURE 4. Absolute error for the 120th-order approximation by optimal VIM when $h = 0.93258$, in example 5.2.

Example 5.3 Consider the following two-dimensional wave equation [4]:

$$\begin{cases} u_{tt} = 2\left(u_{xx} + u_{yy}\right), & & -\infty < x, y < \infty, \quad t > 0, \\ u(x,y,0) = \sin(x)\sin(y), & u_t(x,y,0) = 0, & -\infty < x, y < \infty, \end{cases} \tag{5.10}$$

which admits the solution $u(x,y,t) = \sin(x)\sin(y)\cos(2t)$. Take $(x,y,t) \in (-\infty,\infty)\times(-\infty,\infty)\times[0,50]$ According to the standard VIM we have the following variational iteration formula:

$$\begin{aligned} &u_{n+1}(x,y,t) = u_n(x,y,t) \\ &+ \int_0^t (s-t)\left\{\frac{\partial^2 u_n(x,y,s)}{\partial s^2} - 2\left(\frac{\partial^2 u_n(x,y,s)}{\partial x^2} + \frac{\partial^2 u_n(x,y,s)}{\partial y^2}\right)\right\} ds. \end{aligned} \tag{5.11}$$

Beginning with $u_0(x,y,t) = u(x,y,0) + tu_t(x,y,0) = \sin(x)\sin(y)$, we stop the solution procedure at $u_{120}(x,y,t)$. Figure **5**, is the absolute error of $u_{120}\left(x, 10^8, t\right)$, for $(x,t) \in [-10^8, 10^8] \times [0,50]$, showing that the solution $u_{120}(x,y,t)$ is not valid for large values of x, y and t, of course, the accuracy can be improved if the iteration procedure continues and the exact solution can be obtained when n tends to infinite. Now, using the recursive scheme (**3.1**), we successively have:

$$\begin{aligned} &u_0(x,y,t) = u(x,y,0) + tu_t(x,y,0) = \sin(x)\sin(y), \\ &u_1(x,y,t,h) = \sin(x)\sin(y) - ht^2\sin(x)\sin(y), \end{aligned}$$

and in general,

$$
\begin{aligned}
&u_{n+1}(x,y,t,h) = u_n(x,y,t) \\
&+h\int_0^t (s-t)\left\{\frac{\partial^2 u_n(x,y,s)}{\partial s^2} - 2\left(\frac{\partial^2 u_n(x,y,s)}{\partial x^2} + \frac{\partial^2 u_n(x,y,s)}{\partial y^2}\right)\right\} ds,\ n \geqslant 1.
\end{aligned} \tag{5.12}
$$

In order to find a proper value of h for the approximate solutions (**5.12**), we define the following residual function,

$$r_{120}(x,y,t,h) = \frac{\partial^2 u_{120}(x,y,t,h)}{\partial t^2} - 2\left(\frac{\partial^2 u_{120}(x,y,t,h)}{\partial x^2} + \frac{\partial^2 u_{120}(x,y,t,h)}{\partial y^2}\right),$$

and the following error of residual function,

$$e_{120}(h) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{50} |r_{120}(x,y,t,h)|^2\, dt\, dy\, dx,$$

clearly, suitable value of h, is the global minimum point of $e_{120}(h)$ which we obtained $h = 0.93253$, using Maple software in 5 seconds.. The absolute error of 120th-order approximation of the proposed method for $u_{120}(x,10^8,t)$ in the solution domain $(x,t) \in [-10^8, 10^8] \times [0,50]$, is given in Figure **6**, the accuracy is remarkably improved by the optimal choice of h.

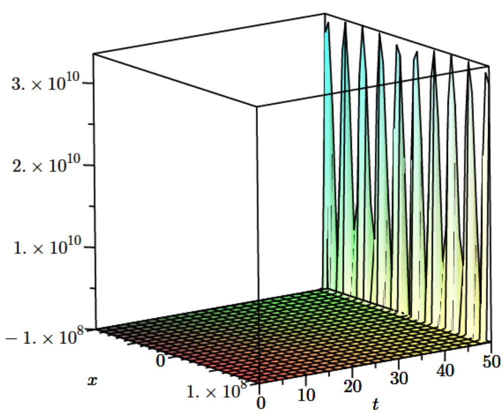

FIGURE 5. Absolute error for the 120th-order approximation by standard VIM for $u_{120}(x,10^8,t)$ in example 5.3.

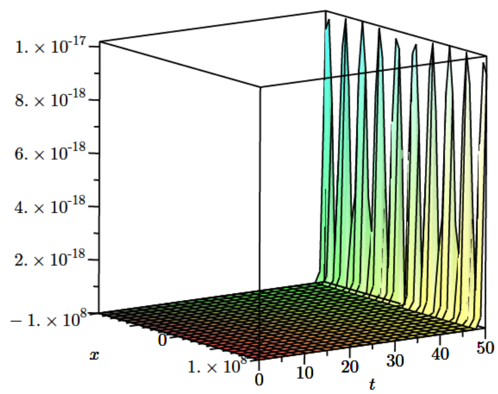

FIGURE 6. Absolute error for the 120th-order approximation by present technique for $u_{120}(x,10^8,t)$ when $h = 0.93253$ in example 5.3.

Conclusion

Many application problems have been successfully solved using the variational iteration process. The focus of this article is on how the propounded approach may manage solutions of wave equations in large domains, while the standard VIM may have difficulty obtaining adequate precision in large domains. It is demonstrated that VIM with an auxiliary parameter is a powerful and efficient tool for wave equations. Comparing the original VIM, our approach is simple to apply and capable of approximating solutions more precisely over a longer interval. In particular, the proposed approach widens the convergence zone. Furthermore, the proposed approach can be conveniently extended to a broad range of nonlinear problems in large domains.

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