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Research Paper

AUTOMATIC CONTINUITY OF SURJECTIVE ALMOST DERIVATIONS ON FRECHET Q-ALGEBRAS

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ABSTRACT. In 1971 R. L. Carpenter proved that every derivation T on a semisimple commutative Frechet algebra Λ with identity is continuous. By relaxing the commutativity assumption on Λ and adding the surjectivity assumption on T, we derive a corresponding continuity result, for a new concept of almost derivations on Frechet algebras in this article. Also, it is further proved that every surjective almost derivation T on non commutative semisimple Frechet Q-algebras Λ with an additional condition on Λ, is continuous. Moreover, an example is provided to illustrate our main result.

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1. Introduction

We provide a brief outline of definitions and known outcomes in this section. For more details, one may refer to [2, 7]. All vector spaces are considered over the complex field, and we assume that all algebras are unital. An Banach algebra Λ is a complete normed algebra, where a normed algebra Λ is an algebra with a norm $||.||$, which also satisfies $||p.q|| \leq ||p||.||q||$, for all $p, q \in \Lambda$. An algebra with a Hausdorff topology is called a topological algebra, if all algebraic operations are jointly continuous. The Jacobson radical $rad(\Lambda)$ of an algebra Λ is the intersection of all maximal right(or left) ideals. An algebra is said to be semisimple, if $rad(\Lambda) = \{0\}$.

Definition 1.1. [2] The spectrum $\sigma_\Lambda(p)$ of an element p of an algebra Λ is the set of all complex numbers γ such that $\gamma.1-p$ is not invertible in Λ. The spectral radius $r_\Lambda(p)$ of an element $p \in \Lambda$ is defined by $r_\Lambda(p) = \sup\{|\gamma| : \gamma \in \sigma_\Lambda(p)\}$.

If $(\Lambda, ||.||)$ is a Banach algebra, then $r_\Lambda(p) = \lim_{n\to\infty} ||p^n||^{\frac{1}{n}}$. Also, for any Banach algebra Λ, we have $rad(\Lambda) = \{p \in \Lambda : r_\Lambda(pq) = 0,$ for every $q \in \Lambda\}$. See ([14], Lemma 1).

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Definition 1.2. Let Λ and Γ be two complete metrizable topological vector spaces. Let $T : \Lambda \to \Gamma$ be a linear map. Then the separating space of T is defined as the set
$S(T) = \{q \in \Gamma : \text{ there exists } (p_n)_{n=1}^{\infty} \text{ in } \Lambda \text{ such that } p_n \to 0 \text{ and } Tp_n \to q\}$.

Also, $S(T)$ is a closed linear subspace of Γ. Moreover, $S(T) = \{0\}$ if and only if T is continuous, because of closed graph theorem. For a proof, see ([2], 5.1.2).

Lemma 1.3. *([14], Lemma 2) Let Λ be a Banach algebra, $P(z)$ a polynomial with coefficients in Λ and $R > 0$. Then*

$$r_{\Lambda}^2(P(1)) \leq \sup_{|z|=R} r_{\Lambda}(P(z)). \sup_{|z|=\frac{1}{R}} r_{\Lambda}(P(z)).$$

A complete metrizable topological algebra is called an F-algebra. A topological algebra Λ is said to be a LMC algebra, if its topology be induced by a separating family of submultiplicative seminorms. A Frechet algebra is a LMC algebra which is also an F-algebra. A Q-algebra is a topological algebra in which the set of all invertible elements is open. A metrizable LMC algebra is written in the form $(\Lambda, (p_n)_{n=1}^{\infty})$, where $(p_n)_{n=1}^{\infty}$ is a separating sequence and each p_n is a submultiplicative seminorm (i.e. $p_n(u.v) \leq p_n(u).p_n(v)$, for all $u, v \in \Lambda$) satisfying $p_n(u) \leq p_{n+1}(u)$, for all $n \in \mathbb{N}$, and $u \in \Lambda$, in which the topology on Λ is induced by the seminorms $p_n, n \in \mathbb{N}$. Also, a sequence (y_k) in the Frechet algebra $(\Lambda, (p_n))$ converges to $y \in \Lambda$ if and only if $p_n(y_k - y) \to 0$, for every $n \in \mathbb{N}$, as $k \to \infty$. In a Frechet Q-algebra, spectral radius of every element is a finite number, see [7]. Every Banach algebra is a Frechet Q-algebra.

Remark 1.4. Let $(\Lambda, (p_n))$ be a Frechet algebra, and Λ_n be the completion of the quotient algebra $\Lambda/ker\ p_n$, with respect to the norm ${p_n}'(y + ker\ p_k) = p_n(y), y \in \Lambda$, then Λ_n is a Banach algebra.

Definition 1.5. [6] Let Λ be an algebra. A linear map $T : \Lambda \to \Lambda$ is called derivation, if $T(\mu.\eta) = \mu.T(\eta) + T(\mu).\eta$, for all $\mu, \eta \in \Lambda$.

Recently T.G. Honary et al introduced the concept of almost multiplicative maps between Frechet algebras in [4]. Next, we introduce almost derivations on Frechet algebras.

Definition 1.6. Let $(\Lambda, (p_n))$ be a Frechet algebra. A linear map $T : \Lambda \to \Lambda$ is called almost derivation, if there are $\epsilon_n \geq 0$ such that $p_n(T(\mu.\eta) - \mu.T(\eta) - T(\mu).\eta) \leq \epsilon_n p_n(\mu)\ p_n(\eta)$; for all $n \in \mathbb{N}$, and $\mu, \eta \in \Lambda$.

Remark 1.7. If $\epsilon_n = 0$, for every n, then almost derivations on Λ turn out to be derivations on Λ, because (p_n) is a separating sequence of seminorms on Λ. Also, every derivation is an almost derivation, for every $\epsilon_n \geq 0$.

A conjecture of Kaplansky[6] can be stated in the following question form. Is every derivation on semisimple Banach algebra continuous?. Kaplansky conjecture was proved by Johnson and Sinclair[5] in 1968. In 1971, R.

L. Carpenter[1] proved that every derivation on a semisimple commutative Frechet algebra with identity is continuous. There are some recent articles [9, 10, 11, 12, 13] for automatic continuity of derivations in the theory of topological algebras.

In this article, we prove that every surjective almost derivation T on a semisimple Frechet Q-algebra $(\Lambda, (p_n))$, with an additional condition on $(\Lambda, (p_n))$, is continuous.

2. main results

Theorem 2.1. *Let $(\Lambda, (p_n))$ be a semisimple Frechet algebra(not necessarily commutative), and Λ_k be the completion of $\Lambda/ker\ p_k$, with respect to the norm ${p_k}'(y + ker\ p_k) = p_k(y), y \in \Lambda$. If $T : \Lambda \to \Lambda$ is a surjective almost derivation such that $r_{\Lambda_k}(Tx + ker\ p_k) \leq p_k(x)$, for all $k \in \mathbb{N}$ and $x \in \Lambda$, then T is continuous.*

Proof. Since closed graph theorem, for proving continuity of T, we have to prove that $b = 0$ for every arbitrary sequence $(\tau_n)_{n=1}^{\infty}$ in Λ such that $\tau_n \to 0$, and such that $T(\tau_n) \to b$. Let us begin with such a sequence $(\tau_n)_{n=1}^{\infty}$ and b.

Since T is onto, there exists $a \in \Lambda$ such that $Ta = b$. We define $P_n(z) = zT\tau_n + T(a - \tau_n)$. Since for each $y \in \Lambda$, $r_{\Lambda_k}(y + ker\ p_k) \leq {p_k}'(y + ker\ p_k) = p_k(y)$, we have

$$\begin{aligned} r_{\Lambda_k}(P_n(z) + ker\ p_k) &\leq p_k(P_n(z)) \\ &\leq |z|p_k(T\tau_n) + p_k(b - T\tau_n). \end{aligned}$$

By hypothesis, we also have

$$\begin{aligned} r_{\Lambda_k}(P_n(z) + ker\ p_k) &= r_{\Lambda_k}(T(z\tau_n + a - \tau_n) + ker\ p_k) \\ &\leq p_k(z\tau_n + a - \tau_n) \\ &\leq |z|p_k(\tau_n) + p_k(a - \tau_n). \end{aligned}$$

By Lemma 1.3, we have

$$\begin{aligned} r^2_{\Lambda_k}(b + ker\ p_k) &= r^2_{\Lambda_k}(P_n(1) + ker\ p_k) \\ &\leq \sup_{|z|=R} r_{\Lambda_k}(P_n(z) + ker\ p_k). \sup_{|z|=\frac{1}{R}} r_{\Lambda_k}(P_n(z) + ker\ p_k) \\ &\leq (Rp_k(\tau_n) + p_k(a - \tau_n))(\frac{1}{R}p_k(T\tau_n) + p_k(b - T\tau_n)), \end{aligned}$$

for every fixed $R > 0$. We fix k and take $n \to \infty$ to obtain

$$r^2_{\Lambda_k}(b + ker\ p_k) \leq p_k(a).\frac{1}{R}p_k(b).$$

Now, let $R \to \infty$ to get $r_{\Lambda_k}(b + ker\ p_k) = 0$, for each k and therefore $r_\Lambda(b) = 0$, because $r_\Lambda(b) = \sup_{k\in\mathbb{N}} r_{\Lambda_k}(b + ker\ p_k)$, see, for example, ([8], Corollary 5.13).

Let $c \in \Lambda$. Since $(\tau_n)_{n=1}^{\infty}$ is a sequence in Λ such that $\tau_n \to 0$, we have $p_k(c.\tau_n) \leq p_k(c).p_k(\tau_n) \to 0$, for all $k \in \mathbb{N}$, as $n \to \infty$. Let $w = T(c)$. Since T is an almost derivation, we have

$$\begin{aligned} p_k(T(c.\tau_n) - c.b) &\leq p_k(T(c.\tau_n) - c.T(\tau_n) - T(c).\tau_n) \\ &\quad + p_k(c.T(\tau_n) + w.\tau_n - c.b) \\ &\leq p_k(T(c.\tau_n) - c.T(\tau_n) - T(c).\tau_n) \\ &\quad + p_k(c.T(\tau_n) - c.b) + p_k(w.\tau_n) \\ &\leq \epsilon_k p_k(c)\, p_k(\tau_n) + p_k(c)\, p_k(T(\tau_n) - b) + p_k(w.\tau_n). \end{aligned}$$

Since $p_k(T(\tau_n) - b) \to 0$, $p_k(\tau_n) \to 0$ and $p_k(w.\tau_n) \leq p_k(w).p_k(\tau_n) \to 0$, for all $k \in \mathbb{N}$, as $n \to \infty$, we have $p_k(T(c.\tau_n) - c.b) \to 0$, for every k, and hence $T(c.\tau_n) \to c.b$, when $c.\tau_n \to 0$. By the same argument mentioned at the beginning of the proof, we have $r_\Lambda(c.b) = 0$. Since $c \in \Lambda$ is arbitrary, we conclude this $b \in rad(\Lambda) = \{0\}$, and this proves the theorem. □

Corollary 2.2. *Let $(\Lambda, (p_n))$ be a semisimple Frechet Q-algebra. If $T : \Lambda \to \Lambda$ is a surjective almost derivation with $r_\Lambda(Ta) \leq r_\Lambda(a)$, for all $a \in \Lambda$. Then T is continuous.*

Proof. We known that $r_\Lambda(x) \leq p_m(x)$, for some $m \in \mathbb{N}$, and for all $x \in \Lambda$, because Λ is a Q-algebra. See, for example, ([3], Theorem 6.18). Since $p_k(x) \leq p_{k+1}(x)$, for all x, and $k \in \mathbb{N}$, we have

$$r_{\Lambda_k}(Tx + ker\ p_k) \leq r_\Lambda(Tx) \leq r_\Lambda(x) \leq p_m(x) \leq p_k(x),$$

for all $k \geq m$ and $x \in \Lambda$. So, by Theorem 2.1, T is continuous. □

Corollary 2.3. *Let Λ be a semisimple Banach algebra. If $T : \Lambda \to \Lambda$ is a surjective almost derivation with $r_\Lambda(Ta) \leq r_\Lambda(a)$, for all $a \in \Lambda$, then T is continuous.*

Example 2.4. Let $(\Lambda, (p_n))$ be a semisimple Frechet Q-algebra. A linear map $T : \Lambda \to \Lambda$ is defined by $T(a) = \beta a$, for all $a \in \Lambda$ where $\beta \in (0, 1]$. Since

$$\begin{aligned} p_n(T(\mu.\eta) - \mu.T(\eta) - T(\mu).\eta) &= p_n(\beta\mu.\eta - \mu.\beta\eta - \beta\mu.\eta) \\ &= p_n(-\beta\mu.\eta) \leq |-\beta| p_n(\mu).p_n(\eta), \end{aligned}$$

for all $\mu, \eta \in \Lambda$, hence T is an almost derivation but not a derivation on $(\Lambda, (p_n))$. Since Λ is a Q-algebra, there exists $k \in \mathbb{N}$ such that $r_\Lambda(a) = \lim_{n\to\infty}(p_k(a^n))^{\frac{1}{n}}$, for all $a \in \Lambda$. See, for example ([3], Theorem 6.18). So

$$r_\Lambda(Ta) = r_\Lambda(\beta a) = \lim_{n\to\infty}(p_k((\beta a)^n))^{\frac{1}{n}} = |\beta| \lim_{n\to\infty}(p_k(a^n))^{\frac{1}{n}} \leq r_\Lambda(a).$$

So, all hypothesis of Corollary 2.2 are satisfied and T is continuous.

3. conclusion

R. L. Carpenter result motivates us to ask an open question: Is every surjective almost derivation on semisimple Frechet algebras continuous? Moreover, a partial answer to this open question is derived in the sense that every surjective almost derivation T on semisimple Frechet Q-algebras Λ, with an additional condition on Λ, is continuous.

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