Mathematical Analysis
& Convex Optimization

Vol. 3 (2022), No. 1, 15–22
https:\\ maco.lu.ac.ir
DOI: 10.52547/maco.3.1.2

Research Paper

EXISTENCE OF AT LEAST ONE NON-TRIVIAL PERIODIC SOLUTION FOR A CLASS OF ORDINARY P-HAMILTONIAN SYSTEMS

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Abstract. Based on recent variational methods for smooth functionals defined on reflexive Banach spaces, We prove the existence of at least one non-trivial solution for a class of p-Hamiltonian systems. Employing one critical point theorem, existence of at least one weak solutions is ensured. This approach is based on variational methods and critical point theory. The technical approach is mainly based on the at least one non -trivial solution critical point theorem of G. Bonanno.

MSC(2010): 34B15; 49J40.
Keywords: Multiple periodic solutions, Hamiltonian systems, Critical point theory, Variational methods.

1. Introduction and Background

Consider the following damped vibration problem

$$\begin{cases} -(|u'|^{p-2}u')' + A(t)|u|^{p-2}u = \lambda \nabla F(t,u) & a.e.\ t \in [0,T], \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{1.1}$$

where $T > 0$, $p > 1$, $A : [0,T] \to \mathbb{R}^{N\times N}$ is a continuous map from the interval $[0,T]$ to the set of N-order symmetric matrices, and there exists a positive constant $\underline{\delta}$ such that $(A(t)|x|^{p-2}x, x) \geq \underline{\delta}\,|x|^p$ for all $x \in \mathbb{R}^N$ and $t \in [0,T]$, $\lambda > 0$, $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ are measurable with respect to t, for all $u \in \mathbb{R}^N$, continuously differentiable in u, for almost every $t \in [0,T]$, satisfies the following standard summability condition:

$$\sup_{|\xi|\leq a} \max\{|F(\cdot,\xi)|,\ |\nabla F(\cdot,\xi)| \in L^1([0,T])\} \tag{1.2}$$

for any $a > 0$. It is clear that if ∇F are assumed to be continuous in $[0,T] \times \mathbb{R}^N$, then the condition (2) is satisfied.

In recent years, the three critical points theorem of Ricceri [11] has widely been used to solve differential equations; see [1, 3, 4, 5, 6, 8, 14]and references therein.

In this note, we prove the existence of at least one non-trivial weak solution for the problme (1.1).

More precisely, our aim is to present overview of some recent results devoted to the study of problme (1.1). For an overview on this subject, we refer the reader to the papers [1, 9, 12].

Date: Received: January 27, 2022, Accepted: May 8, 2022.
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For basic notation and definitions, we refer the reader [10, 13, 14]. special cases of problme has been studied in [6, 13] for existence and multiplicity of solutions for damped vibration problems based on variational methods and critial point theory.

2. Preliminaries

Our main tool is the Ricceri's variational principle [11, Theorem 2.5] as given in [2, Theorem 5.1] which is recalled below (see also [2, Proposition 2.1].

For a given non-empty set X, and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define the following functions

$$\beta(r_1, r_2) = \inf_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)},$$

$$\rho(r_1, r_2) = \sup_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(]-\infty, r_1[)} \Psi(u)}{\Phi(v) - r_1}$$

for all $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$.

Theorem 2.1. [2, Theorem 5.1] *Let X be a real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that there are $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho(r_1, r_2).$$

Then, setting $I_\lambda := \Phi - \lambda\Psi$, for each $\lambda \in]\frac{1}{\rho(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}[$ there is $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ such that

$I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ $\forall u \in \Phi^{-1}(]r_1, r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

We assume that the matrix A satisfies the following conditions:

(b1) $A(t) = (a_{kl}(t))$, $k = 1, \ldots, N$, $l = 1, \ldots, N$, is a symmetric matrix with $a_{kl} \in L^\infty[0, T]$ for any $t \in [0, T]$

(b2) there exists a positive constant $\underline{\delta}$ such that $(A(t)|x|^{p-2}x, x) \ge \underline{\delta}\,|x|^p$ for all $x \in \mathbb{R}^N$ and $t \in [0, T]$, where $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^N$ and in the other hand we know that $((A(t)|x|^{p-2}x, x) \le \bar{\delta}\,|x|^p$ for any $x \in \mathbb{R}^N$ and for every $t \in [0, T]$ where $\bar{\delta} \le \sum_{i,j=1}^{N} \|a_{ij}\|_\infty$(for more details, see[14]) .

Let us recall some basic concepts. Denote

$$X = \{u : [0, T] \to \mathbb{R}^N |\ u \text{ is absolutely continuous}, \ \ u(0) = u(T), \ \ \dot{u} \in L^p([0, T], \mathbb{R}^N)\}$$

The corresponding norm is defined by

$$\|u\|_X = \left(\int_0^T (|\dot{u}(t)|^p + |u(t)|^p) dt \right)^{\frac{1}{p}}, \ \forall\ u \in X.$$

We define

$$\|u\| = \left(\int_0^T |u'(t)|^P dt + \int_0^T (A(t)|u(t)|^{p-2}u(t), u(t))dt \right)^{\dfrac{1}{p}}.$$

Clearly, the norm $\|\cdot\|$ is equivalent to the norm $\|\cdot\|_X$.

Since $(X, \|\cdot\|)$ is compactly embedded in $C([0,T], \mathbb{R}^N)$ (see [10]), for every $q = \frac{p}{p-1}$, there exists a positive constant

$$c = \sqrt[q]{2} \max\{T^{\frac{1}{q}}, T^{\frac{-1}{p}}\}(\min\{1, \underline{\delta}\})^{\frac{-1}{p}}$$

such that

$$\|u\|_\infty \leq c \parallel u \parallel, \tag{2.1}$$

where $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$, for all $x \in \mathbb{R}^N$ and $t \in [0,T]$.

We mean by a (weak) solution of problem (1.1), any function $u \in X$ such that

$$\int_0^T (|u'(t)|^{p-2}u'(t), v'(t))dt + \int_0^T (A(t)|u(t)|^{p-2}u(t), v(t))dt$$

$$-\lambda \int_0^T (\nabla F(t, u(t)), v(t))dt = 0$$

for every $v \in X$.

3. MAIN RESULTS

We present our main result as follows.

Given a non-negative constant θ and $0 \neq x_0 \in \mathbb{R}^N$ such that

$$\theta \neq c|x_0|(\bar{\delta}T)^{\frac{1}{p}},$$

Put

$$a(\theta, x_0) := \frac{\int_0^T \sup_{|\xi|\leq\theta} F(t,\xi)dt - \int_0^T F(t,x_0)dt}{\theta^p - c^p|x_0|^p\bar{\delta}\ T}$$

Now, we formulate our main result.

Theorem 3.1. *Assume that there exist a non-negative constant θ_1 and positive constant θ_2 and given $0 \neq x_0 \in \mathbb{R}^N$ with $|x_0| \in \left] \dfrac{\theta_1}{c(\underline{\delta}T)^{\frac{1}{p}}}, \dfrac{\theta_2}{c(\bar{\delta}T)^{\frac{1}{p}}} \right[$ such that*

(I) $a(\theta_2, x_0) < a(\theta_1, x_0)$;

Then for any $\lambda \in \left] \dfrac{1}{pc^p} \dfrac{1}{a(\theta_1, x_0)},\ \dfrac{1}{pc^p} \dfrac{1}{a(\theta_2, x_0)} \right[$

the problem (1.1) admits at least one non-trivial weak solution $u \in X$ such that $\dfrac{\theta_1}{c} < ||u|| < \dfrac{\theta_2}{c}$.

Proof. In order to apply Theorem 2.1 to our problem, we introduce the functionals $\Phi,\ \Psi : X \to \mathbb{R}$ defined as follows

$$\Phi(u) = \frac{1}{p}\|u\|^p, \tag{3.1}$$

and

$$\Psi(u) = \int_0^T (F(t, u(t))dt \tag{3.2}$$

for every $u \in X$. It is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_0^T (\nabla F(t, u(t)), v(t))dt,$$

for every $v \in X$ as well as is sequentially weakly upper semicontinuous. Furthermore, $\Psi' : X \to X^*$ is a compact operator. Indeed, it is enough to show that Ψ' is strongly continuous on X. For this end, for fixed $u \in X$, let $u_n \to u$ weakly in X as $n \to \infty$, then u_n converges uniformly to u on $[0, T]$ as $n \to \infty$, see [10]. Since F is continuously differentiable in u, for almost every $t \in [0, T]$ hence ∇F is continuous in $\mathbb{R}^{\mathbb{N}}$ for every $t \in [0, T]$, so

$$\nabla F(t, u_n) \to \nabla F(t, u), \qquad as \qquad n \to \infty.$$

Hence according to above result and assumption (2) we have, $\Psi'(u_n) \to \Psi'(u)$ as $n \to \infty$. Thus we proved that Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator by Proposition 26.2 of [15].

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Moreover, Φ is continuously differentiable whose differential at the point $u \in X$ is

$$\Phi'(u)(v) = \int_0^T (|u'(t)|^{p-2}u'(t), v'(t)) + (A(t)|u|^{p-2}u(t), v(t))dt,$$

for every $v \in X$, while Proposition 2.4 of [7] gives that Φ' admits a continuous inverse on X^*. Furthermore, Φ is sequentially weakly lower semicontinuous.

Clearly, the weak solutions of the problem (1.1) are exactly the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$.

Now , put

$$r_1 := \frac{1}{p}(\frac{\theta_1}{c})^p, \qquad r_2 := \frac{1}{p}(\frac{\theta_2}{c})^p, \quad \text{and} \qquad w(t) := x_0 \qquad \text{for all } t \in [0, T].$$

We clearly observe that $w \in X$ and

$$\begin{aligned} \frac{1}{p}|x_0|^p \underline{\delta} T &\leq \Phi(w) = \frac{1}{p}\|x_0\|^p = \frac{1}{p}\int_0^T (A(t)|x_0|^{p-2}x_0, x_0)dt \\ &\leq \frac{1}{p}|x_0|^p \bar{\delta} T. \end{aligned}$$

so from the conditions

$$\theta_1 < |x_0|c(\underline{\delta}T)^{\frac{1}{p}} \qquad and \qquad |x_0|c(\bar{\delta}T)^{\frac{1}{p}} < \theta_2$$

we get

$$r_1 < \Phi(w) < r_2.$$

From the fact $\Phi(u) = \frac{1}{p}\|u\|^p$ for each $u \in X$ and taking (2.1) into account, we see that

$$\begin{aligned}\Phi^{-1}(]-\infty, r_2[) &= \{u \in X;\ \Phi(u) < r_2\}\\ &\subseteq \left\{u \in X;\ \frac{1}{p}\|u\|^p < r_2\right\}\\ &\subseteq \{u \in X;\ |u(t)| \leq \theta_2 \text{ for each } t \in [0,T]\},\end{aligned}$$

and it follows that

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) &= \sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \int_0^T F(t,u(t))dt\\ &\leq \int_0^T \sup_{|\xi|\leq\theta_2} F(t,\xi)dt.\end{aligned}$$

Therefore, one has

$$\begin{aligned}\beta(r_1,r_2) &\leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) - \Psi(w)}{r_2 - \Phi(w)}\\ &\leq \frac{\int_0^T \sup_{|\xi|\leq\theta_2} F(t,\xi)dt - \Psi(w)}{r_2 - \Phi(w)}\\ &\leq \frac{\int_0^T \sup_{|\xi|\leq\theta_2} F(t,\xi)dt - \int_0^T F(t,x_0)dt}{\frac{1}{p}(\frac{\theta_2}{c})^p - \frac{1}{p}|x_0|^p\bar{\delta}T}\\ &= \frac{pc^p(\int_0^T \sup_{|\xi|\leq\theta_2} F(t,\xi)dt - \int_0^T F(t,x_0)dt)}{\theta_2^p - c^p|x_0|^p\bar{\delta}T}\\ &= pc^p a(\theta_2, x_0).\end{aligned}$$

On the other hand, arguing as before, one has

$$\begin{aligned}\rho(r_1,r_2) &\geq \frac{\Psi(w) - \sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{\Phi(w) - r_1}\\ &\geq \frac{\Psi(w) - \int_0^T \sup_{|\xi|\leq\theta_1} F(t,\xi)dt}{\Phi(w) - r_1}\\ &\geq \frac{\int_0^T F(t,x_0)dt - \int_0^T \sup_{|\xi|\leq\theta_1} F(t,\xi)dt}{\frac{1}{p}|x_0|^p\bar{\delta}T - \frac{1}{p}(\frac{\theta_1}{c})^p}\\ &= \frac{pc^p(\int_0^T F(t,x_0)dt - \int_0^T \sup_{|\xi|\leq\theta_1} F(t,\xi)dx)}{c^p|x_0|^p\bar{\delta}T - \theta_1^p}\\ &= pc^p a(\theta_1, x_0).\end{aligned}$$

Hence, from assumption (I), one has $\beta(r_1,r_2) < \rho(r_1,r_2)$. Therefore,employing Theorem 2.1, for each $\lambda \in \left]\frac{1}{pc^p}\frac{1}{a(\theta_1,x_0)}, \frac{1}{pc^p}\frac{1}{a(\theta_2,x_0)}\right[$, the functional $\Phi - \lambda\Psi$ admits at least one critical point $u \in X$ such that $r_1 < \Phi(u) < r_2$, that is $\frac{\theta_1}{c} < \|u\| < \frac{\theta_2}{c}$. □

Now, we point out an immediate consequence of Theorem 3.1.

Theorem 3.2. *Assume that there exist a positive constant* θ *and given* $0 \neq x_0 \in \mathbb{R}^N$ *with*

$$|x_0| c\big(\bar{\delta} T\big)^{\frac{1}{p}} < \theta$$

such that

(I) $F(t, 0) = 0$ *for a.e.* $t \in [0, T]$

(II) $\dfrac{\int_0^T \sup_{|\xi| \leq \theta} \; F(t, \xi) dt}{\theta^P} < \big(\dfrac{1}{c^P |x_0|^P \bar{\delta} T}\big) \int_0^T F(t, x_0) dt$

Then, for every $\lambda \in \Big]\dfrac{1}{p} \dfrac{|x_0|^p \bar{\delta} T}{\int_0^T F(t, x_0) dt}, \dfrac{1}{pc^p} \dfrac{\theta^p}{\int_0^T \sup_{|\xi| \leq \theta} F(t, \xi) dt}\Big[$ *the problem* (1.1) *admits at least one non-trivial solution* $u \in X$ *such that* $\|u\|_\infty < \theta$.

Proof. The conclusion follows from Theorem 3.1, by taking $\theta_1 = 0$ and $\theta_2 = \theta$. Indeed, owing to assumptions (I), (II), one has

$$\begin{aligned} a(\theta, x_0) : \; &= \frac{\int_0^T \sup_{|\xi| \leq \theta} F(t, \xi) dt - \int_0^T F(t, x_0) dt}{\theta^P - c^P |x_0|^P \bar{\delta} T} \\ &< \frac{\big(\dfrac{\theta^p}{c^p |x_0|^p \bar{\delta} T} - 1\big) \int_0^T F(t, x_0) dt}{\theta^p - c^p |x_0|^p \bar{\delta} T} \\ &= \frac{\int_0^T F(t, x_0) dt}{c^p |x_0|^p \bar{\delta} T} = a(0, x_0). \end{aligned}$$

Hence, taking assumption (2.1) into account, Theorem 3.1 yields the conclusion. □

Now we present the following example to illustrate the result.

Example 3.3. Let us define $F(t, x) = e^t(1 + |x|^2)$ for all $t \in [0, 1]$ and $x \in \mathbb{R}^3$. Then for every $\lambda \in].1126, .1488[$ the problem

$$\left\{ \begin{array}{ll} -(|u'|u')' + |u|u = \lambda \nabla F(t, u) & a.e. \;\; t \in [0, 1], \\ u(0) - u(1) = u'(0) - u'(1) = 0. & \end{array} \right. \tag{3.3}$$

admits at least one non-trivial weak solutions $u \in X$ such that $||u|| < 1.8898$. In fact, we can apply Theorem 3.1 where $A(t) = I_{3\times 3}, \bar{\delta} = \underline{\delta} = 1$, $x_0 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \in \mathbb{R}^3$, $\theta_1 = 0, \theta_2 = 3$.

A consequence of Theorem 3.2 is the following existence result.

Theorem 3.4. *Suppose that assumptions* (b1) *and* (b2) *hold. Let* $b \in L^1([0,T])$ *such that* $b(t) \geq 0$ *a.e.* $t \in [0,T]$ *and* $b \not\equiv 0$, $G \in C^1(\mathbb{R}^N, \mathbb{R})$ *such that* $G(0,\dots,0) = 0$ *and*

$$\lim_{x\to 0^+} \frac{\max\limits_{|\xi|\leq x} G(\xi)}{|x|^p} = +\infty. \tag{3.4}$$

Then, for each $\lambda \in (0,\ \lambda')$, *where* $\lambda' := \frac{1}{pc^p \int_0^T b(t)dt} \sup_{\theta>0} \frac{\theta^p}{\max_{|\xi|\leq\theta} G(\xi)}$ *the problem*

$$\begin{cases} -(|u'|^{p-2}u')' + A(t)|u|^{p-2}u = \lambda b(t)\nabla G(u(t)) & a.e.\ t \in [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{cases} \tag{3.5}$$

admits at least a nontrivial periodic solution.

Proof. For fixed $\lambda \in (0,\ \lambda')$, there exists a positive constant θ such that

$$\lambda < \frac{1}{pc^p \int_0^T b(t)dt} \frac{\theta^p}{\max_{|\xi|\leq\theta} G(\xi)}.$$

Moreover, using (3.4) we can choose point x_0 satisfying $|x_0| < \frac{\theta}{c(\bar{\delta}T)^{\frac{1}{p}}}$ such that

$$\frac{\bar{\delta}T}{p\lambda \int_0^T b(t)dt} < \frac{G(x_0)}{|x_0|^p}.$$

Hence, Theorem 3.2 leads to the conclusion. □

4. ACKNOWLEDGEMENTS

The authors are very thankful for the many helpful suggestions and corrections given by the refrees who reviewed this paper.

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